10/14/21

Announcements

HW 4 due 10/21
Quiz 4 Monday 10/17 Newton's laws in rot. frame, Coriolis force

Last time: tides

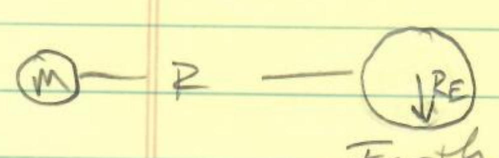
We found just from applying $\vec{F} = \frac{-G m_1 m_2}{r^2}\hat{r}$ to Earth + oceans

effective potential near Earth $r \ll R$

$$\Phi = \frac{U}{m} = -\frac{G M_e}{r} - \frac{G M r^2}{2R^3}\left(3\sin^2\theta\cos^2\phi - 1\right)$$

Water surface is <u>equipotential</u>

$$\Rightarrow \quad h(\theta,\phi) = r - R_e = \frac{R_e}{2}\left(\frac{M_m}{M_e}\right)\left(\frac{R_e}{R}\right)^3\left(\sin^2\theta\cos^2\phi - 1\right)$$



$$\Delta h = \frac{3R_e}{2}\left(\frac{M_m}{M_e}\right)\left(\frac{R_e}{R}\right)^3 \sin^2\theta$$

$$\simeq 0.56\,\text{m}$$
max average

- tides max at equator $\theta = 90^\circ$, 0 at poles (Not quite true because of Earth's tilt)

- Note our calculation applies also to Earth–sun system! $\Delta h \sim \frac{3R_e}{2}\left(\frac{M_s}{M_e}\right)\left(\frac{R_e}{R_{e\text{-}s}}\right)^3 \sin^2\theta$

Sun factor 2 weaker
not negligible!

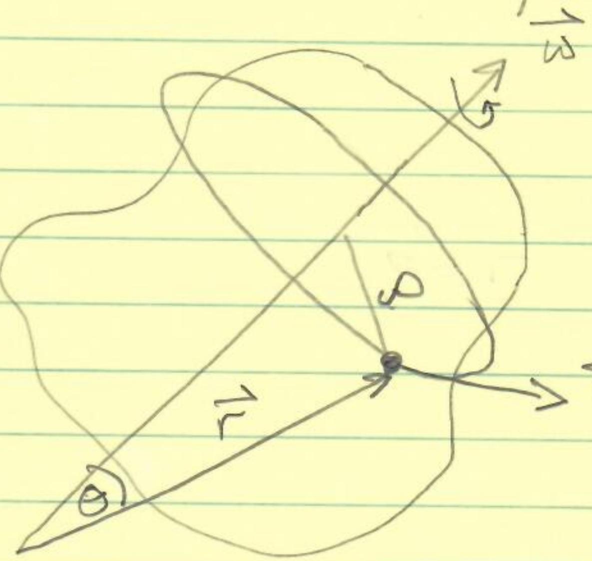
$$\simeq 0.25\,\text{m}$$

Newton's laws in rotating frame
e.g. Earth!

Angular velocity vector $\vec{\omega} = \omega\hat{\omega}$
book calls it $\vec{\omega} = \omega\vec{u}$ unit vector

direction of $\vec{\omega}$ determined by right hand rule.

- relation between rotating rigid body $\vec{\omega}$ and linear velocity $\vec{v}$ of some point:



$\rho = r\sin\theta$

$\vec{v} = \vec{\omega} \times \vec{r}$

✱ This is for the vector $\vec{r}$, $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$
but it holds for any vector fixed in the frame of the body!
e.g. $\vec{e}$ is rotating vector in body

$$\left.\frac{d\vec{e}}{dt}\right|_{S_0} = \vec{\omega} \times \vec{e}$$

calculated in inertial frame

— addition of angular velocities

Frames S_0 – inertial and S – rotating

So S_0 and S have relative angular velocity $\vec{\Omega}$

Recall: if two frames 2, 1 have relative velocity $\vec{v}_{21}$, and body 3 has rel. vel. $\vec{v}_{32}$ relative to 2,

$$\vec{v}_{31} = \vec{v}_{32} + \vec{v}_{21}$$

$$\Rightarrow \vec{\omega}_{31} \times \vec{r} = \vec{\omega}_{32} \times \vec{r} + \vec{\omega}_{21} \times \vec{r}$$

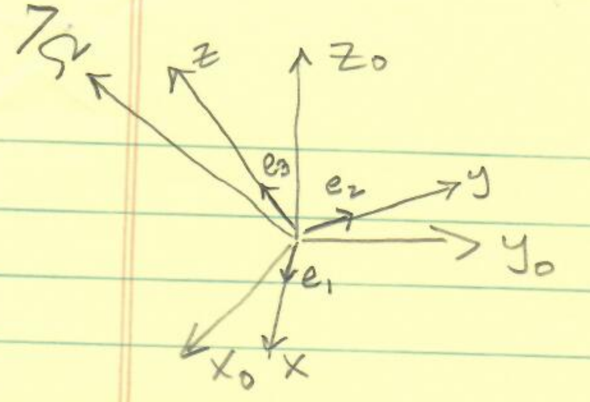
$$= \vec{\omega}_{32} \times \vec{r} + \vec{\Omega} \times \vec{r}$$

True $\forall \vec{r} \Rightarrow \boxed{\vec{\omega}_{31} = \vec{\omega}_{32} + \vec{\Omega}}$

— Time derivatives in rotating frame

Consider arbitrary vector $\vec{Q}$ varying in time

Question: how are $\left.\frac{d\vec{Q}}{dt}\right|_{S_0}$ and $\left.\frac{d\vec{Q}}{dt}\right|_{S}$ related?



note $\hat{z}$ doesn't necessarily point along $\vec{\Omega}$

Express $\vec{Q}$ in terms of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ fixed in S (not S_0)

$$\vec{Q} = Q_1\hat{e}_1 + Q_2\hat{e}_2 + Q_3\hat{e}_3 = \sum_i Q_i\hat{e}_i$$

In S, $\hat{e}_i$ are <u>constant</u> in time
Therefore

$$\left.\frac{dQ}{dt}\right|_S = \sum_i \frac{dQ_i}{dt}\hat{e}_i$$

* Vector $\vec{Q}$ same in both frames => components Q_i also same. An observer in S_0 sees $\hat{e}_i$ rotating with time, but says Q_i measured with respect to these directions are the same.

$$\Rightarrow \frac{dQ_i}{dt} \text{ is same for both } S + S_0$$

In S_0 $\hat{e}_i = \hat{e}_i(t)$, so

$$\left.\frac{d\vec{Q}}{dt}\right|_{S_0} = \sum_i \frac{dQ_i}{dt}\hat{e}_i + \sum_i Q_i \left.\frac{d\hat{e}_i}{dt}\right|_{S_0}$$

but we said for any vector $\vec{e}$ fixed in S,

$$\left.\frac{d\vec{e}}{dt}\right|_{S_0} = \vec{\omega} \times \vec{e}$$

so

$$\boxed{\begin{aligned}\left.\frac{d\vec{Q}}{dt}\right|_{S_0} &= \sum_i \frac{dQ_i}{dt}\hat{e}_i + \sum_i Q_i \vec{\Omega} \times \hat{e}_i \\ &= \left.\frac{d\vec{Q}}{dt}\right|_S + \vec{\Omega} \times \vec{Q}\end{aligned}}$$

Apply now to Newton's laws:

In inertial frame we know $m \left.\frac{d^2\vec{r}}{dt^2}\right|_{S_0} = \vec{F}$

In rotating frame,

$$\left.\frac{d\vec{r}}{dt}\right|_{S_0} = \left.\frac{d\vec{r}}{dt}\right|_S + \vec{\Omega} \times \vec{r}$$

$\frac{d}{dt}$ again: $\left.\frac{d^2\vec{r}}{dt^2}\right|_{S_0} = \left.\frac{d}{dt}\right|_{S_0} \left(\frac{d\vec{r}}{dt}\right)_{S_0}$

$$= \left.\frac{d}{dt}\right|_{S_0} \left[\left.\frac{d\vec{r}}{dt}\right|_S + \Omega \times \vec{r} \right]$$

$$= \left.\frac{d}{dt}\right|_S [\ \ ] + \vec{\Omega} \times [\ \ ]$$

* Notation: in this context, use dot $\dot{\vec{Q}}$ to mean $\left.\frac{d\vec{Q}}{dt}\right|_S$

$$\Rightarrow \left.\frac{d^2 r}{dt^2}\right|_{S_0} = \ddot{\vec{r}} + \vec{\Omega} \times \dot{\vec{r}} + \vec{\Omega} \times \dot{\vec{r}} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\underbrace{m \left.\frac{d^2 r}{dt^2}\right|_{S_0}}_{\vec{F}} = m\ddot{\vec{r}} + m 2\vec{\Omega} \times \dot{\vec{r}} + m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

Coriolis Force　　Centrifugal Force

$$\boxed{m\ddot{\vec{r}} = \vec{F} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}}$$

inertial frame

rotating frame