10/26/22

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## Announcements

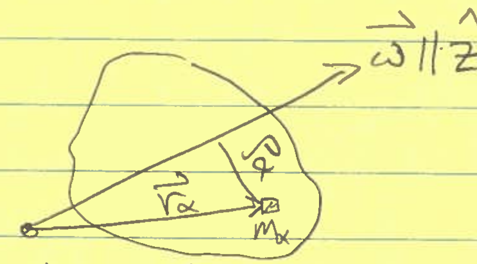
HW 5 due Oct 31
Skip sections 10.7 – 10.10

## Last time

### Rotation of rigid body around fixed axis $\vec{\omega}$

Angular momentum due to rotation (not CM motion)

$\vec{v} = \dot{\vec{r}} = \vec{\omega} \times \vec{r}$ $\qquad \vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$




$$L_z = \sum_\alpha m_\alpha \rho_\alpha^2 \omega = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)\omega = I_z \omega$$

KE of motion wrt CM $= \frac{1}{2}\sum_\alpha m_\alpha (\rho_\alpha \omega)^2 = \frac{1}{2} I_z \omega^2$

$=$ KE of rotation for rigid body

Other $\vec{L}$ components:

$$L_x = -\left(\sum_\alpha m_\alpha x_\alpha z_\alpha\right)\omega \equiv I_{xz}\omega, \text{ etc}$$

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha = I_{yx} \qquad I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha = I_{zy}$$

So $\vec{L}$ and $\vec{\omega}$ need not be $\parallel$!

$\boxed{\text{Quiz 5}}$

Example – uniform thin rod, $\lambda$ mass/length $\quad dm = \lambda\, dx$

about center: $I_{zz} = \int dm\, x^2 = \lambda \int_{-L/2}^{L/2} dx\, x^2 = \lambda \frac{x^3}{3}\Big|_{-L/2}^{L/2} = \frac{\lambda L^3}{12} = \frac{ML^2}{12}$

about end: $I_{zz} = \lambda \int_0^L dx\, x^2 = \frac{ML^2}{3}$

$\parallel$-axis theorem from Physics I:

$$\boxed{I = I_{cm} + Mh^2}$$

$h$ = displacement of rot. axis from CM

Check for our rod: $h = L/2$

$$\frac{ML^2}{3} = \frac{ML^2}{12} + \frac{ML^2}{4} \quad \checkmark$$

Rotation about arb. axis – Inertia Tensor

$$\vec{\omega} = \omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}$$

Evaluate $\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$

$$= \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

ang. mom. of CM neglected!

find $L_i = \sum_{j=1}^{3} I_{ij}\, \omega_j$

e.g. $L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

etc.

Elements of $I_{ij}$

$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)$$

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha$$

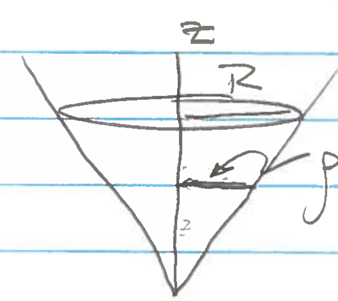
$$\vec{L} = \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \underbrace{\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}}_{\underline{\underline{I}}} \underbrace{\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{\vec{\omega}}$$

$$\vec{L} = \underline{\underline{I}}\,\vec{\omega}$$

Note $\underline{\underline{I}}$ is <u>symmetric</u>

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha = -\sum_\alpha m_\alpha y_\alpha x_\alpha = I_{yx}$$

For solids of rotation, off-diagonal terms will often be zero if axes are chosen carefully.




<u>Example 1: cone</u>

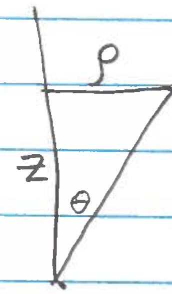
$$I_{xy} = -\frac{M}{V}\int_{\text{cone}} dx\,dy\,dz\; xy = 0$$

by symmetry

$$\tan\theta = \frac{\rho}{z} = \frac{R}{h}$$

$$\rho = zR/h$$

since for each $y, z$, integral over $x$ is symmetric, $\int_{-x_0}^{x_0} dx\, x = 0$




Diagonal elements

$$I_{zz} = \frac{M}{V}\int_0^h dz \int_0^{2\pi} d\phi \int_0^{Rz/h} \rho\, d\rho\; \rho^2$$

$$\boxed{dV = dz\, d\phi\, \rho\, d\rho}$$

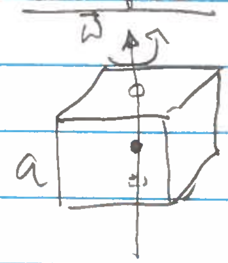
$$= \frac{M}{V}\, 2\pi \int_0^h dz\, \frac{(Rz/h)^4}{4}$$

$$= \frac{M}{V}\, \frac{2\pi}{4} \left(\frac{R}{h}\right)^4 \frac{h^5}{5}$$

$$= \frac{M}{\frac{1}{3}\cancel{\pi} R^2 \cancel{h}}\, \frac{\cancel{\pi} R^4 \cancel{h}}{10} = \frac{3}{10} M R^2$$

$$I_{xx} = \frac{M}{V}\, 2\pi \int_V dV\,(y^2+z^2) \quad \text{harder}$$

Example 2: cube w/ axis through center of 2 faces + CM




$$\left(\frac{M}{a^3}\right)\int_{-a/2}^{a/2} dz \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy\, \left(x^2+y^2\right)$$

x + y equivalent

$$= \frac{M}{a^3} \cdot a^2 \cdot 2 \cdot \left.\frac{x^3}{3}\right|_{-a/2}^{a/2} = \frac{Ma^2}{6}$$

By symmetry $I_{xx} = I_{yy} = I_{zz}$

e.g. $I_{xx} = \rho \int dV \, (y^2 + z^2) = I_{zz}$

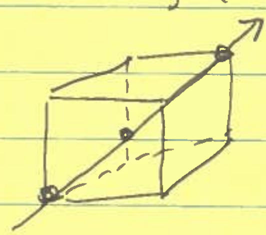
Also: $I_{ij}$, $i \neq j$ are zero!

$$I_{xy} = -\rho \int_{-a/2}^{a/2} dz \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \; x\,y = 0$$

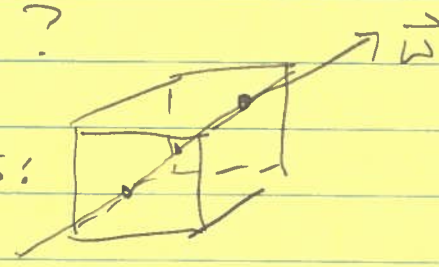
So $\underline{\underline{I}} = \frac{Ma^2}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in these coordinates

Q: what about rotations about other axes through CM?

verticis:



edges:




Remarkable result: $\underline{I}$ is the same for any axis through CM <u>for a cube</u> (only)