10/28/22

Announcements

HW5 due Monday
Not responsible for 10.7 – 10.10

Last time

ang. mom. $\vec{L} = \underline{I}\,\vec{\omega}$ or $L_i = \sum_j I_{ij}\,\omega_j$

i, j run over 1, 2, 3 or x, y, z

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} r_\alpha^2 - r_{\alpha i} r_{\alpha j} \right)$$

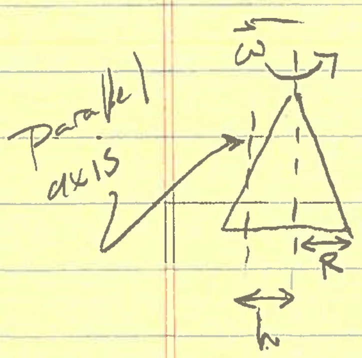
e.g. $i = j = x$ $\quad I_{xx} = \sum_\alpha m_\alpha \left(r_\alpha^2 - x_\alpha^2 \right)$

$$= \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2 \right)$$

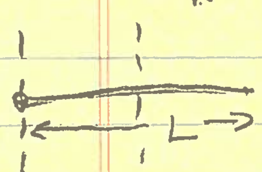
$i = x, j = y$ $\quad I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha$

Parallel axis theorem

$$I = I_{cm} + M h^2$$



- E.g. cone, we found $I_{zz} = \frac{3MR^2}{10}$ for axis thru cM



- Rod: $I_{cm} = \frac{ML^2}{12}$; $I_{end} = \frac{ML^2}{3}$

Example 2; cube w/ axis through center of 2 faces, and CM

①

$$I = \frac{M}{a^3}\int_{-a/2}^{a/2} dz \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \,(x^2+y^2)$$

$$= \frac{m}{a^3}\cdot a^2 \cdot 2 \cdot \frac{x^3}{3}\Big|_{-a/2}^{a/2} = \frac{Ma^2}{6}$$

By symmetry $I_{xx} = I_{yy} = I_{zz}$

e.g. $I_{xx} = \rho \int dV \,(y^2+z^2) = I_{zz} = \frac{Ma^2}{6}$

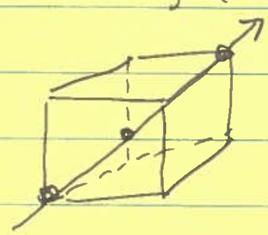
Also: I_{ij}, $i \neq j$ are zero!

$$I_{xy} = -\rho \int_{-a/2}^{a/2} dz \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \; x\,y = 0$$

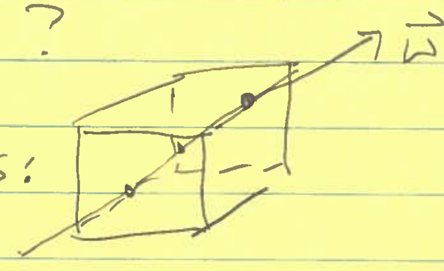
So $\underline{\underline{I}} = \frac{Ma^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in these coordinates

Q: what about rotations about other axes through CM?

verticies:



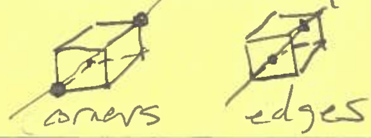
edges:



Remarkable result: $\underline{\underline{I}}$ is the same for any axis through CM for a cube (only)

Experiment torsional oscillations frequency $\omega_0 = \sqrt{\kappa/I}$ same for any axis thru CM

Strange result: $\underline{I}$ is same for any axis through CM.



Reason: $\underline{I}$ doesn't change under rotations of $\vec{\omega}$ to new direction — equivalent to rotation of coordinate axes, accomplished by rotation matrix $\mathcal{O}$, with inverse (reverse rotation) $\mathcal{O}^{-1}$.

Suppose we rotate $\vec{\omega}$ to point to new direction relative to cube axes:

$$\vec{\omega} \rightarrow \mathcal{O}\vec{\omega} = \vec{\omega}\,'$$

we want to calculate $\vec{L}\,'$, $\underline{I}\,'$ in new coordinates

$$\vec{L}\,' = \mathcal{O}\vec{L} = \mathcal{O}\,\underline{I}\,\vec{\omega} = \mathcal{O}\,\underline{I}\,\underbrace{\mathcal{O}^{-1}\mathcal{O}}_{\mathbb{1}}\vec{\omega}$$

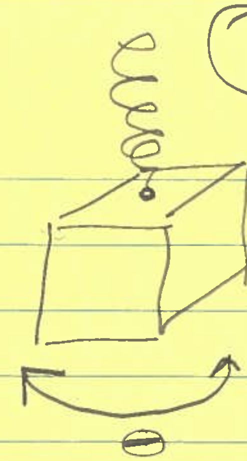
$$= \underline{I}\,'\,\vec{\omega}\,'$$

But for $\underline{I} \propto \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$, $\mathcal{O}\,\underline{I}\,\mathcal{O}^{-1} \propto \mathcal{O}\mathcal{O}^{-1} = \mathbb{1}$

so $\underline{I}$ doesn't change!

Consequence (demo):

frequencies of torsional oscillations same for all axes through CM



Why? Dynamics

torque $\quad \vec{\tau} = \underline{I}\,\vec{\alpha}$

linear restoring torque

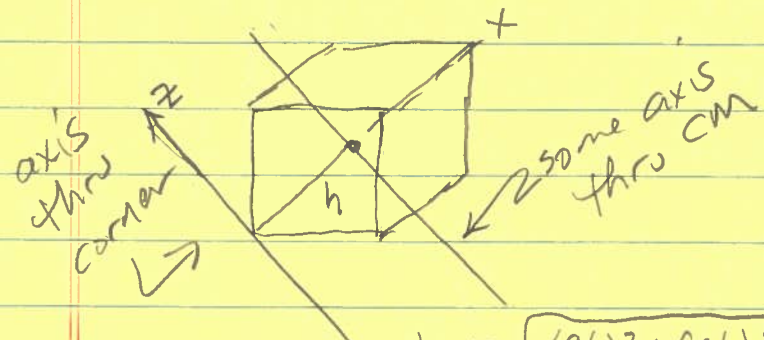
$$\tau = -\kappa\theta \qquad \alpha = \ddot{\theta}$$

$$\ddot{\theta} = -\frac{\kappa}{I}\,\theta$$

$$\Rightarrow \qquad \omega_0 = \sqrt{\kappa/I}$$

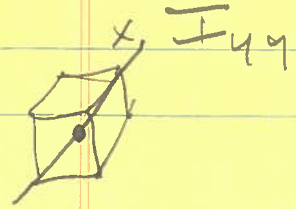
But now we know $\underline{I}$ same for any direction of $\vec{\alpha} \Rightarrow$ all ω_0 same

Book spends 3 pp. algebra to calculate $\underline{I}$ for cube rotating around a corner (Fig. 10.5, Example 10.4).
But we can do it in 2 lines w/ ∥ axis thm

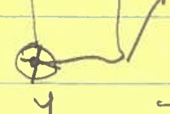


$$I_{zz} = \frac{1}{6}Ma^2 + Mh^2$$
$$= {}^{11}/_{12}\, a^2$$

$$h = \sqrt{(a/2)^2 + (a/2)^2 + (a/2)^2} = \frac{\sqrt{3}}{2}a$$



I_{yy}



must be the same ${}^{11}/_{12}\, a^2$ ✓

I_{xx} passes thru CM $\Rightarrow \frac{1}{6}a^2$