10/31/22 ⑥

Announcements

HW6 posted due 11/14
Read Ch. 11

Last time

- cube has inertia tensor for rotations around axes x, y, z that pass through CM that is $\propto \underline{\underline{1}}$ (unit tensor)

$\Rightarrow$ rotations $\theta \underline{\underline{I}} \theta^{-1}$ preserve $\underline{\underline{I}} \propto \underline{\underline{1}}$

$\Rightarrow$ frequency of small torsional oscillations

$$\ddot{\theta} = -\frac{\kappa}{I}\theta \Rightarrow \omega_0^2 = \frac{\kappa}{I}$$

independent of the orientation of the axis!

Cube attached to spring at a) face; b) corner; c) middle of edge

$\Rightarrow$ all ω_0 same ?

Rot. matrices

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \cos\beta\sin\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma \\ -\sin\beta & \sin\alpha\cos\beta & \cos\alpha\cos\beta \end{bmatrix}$$

2D:

3D: α, β, γ are Euler angles

Principal axes of inertia

If for given object, $\vec{L} \parallel \vec{\omega}$ for rotation around some axis $\vec{\omega}$, that axis is called **principal axis**. Must be that

$$\vec{L} = \lambda \vec{\omega} \quad \text{for that axis}$$

In cube + cone examples, intuition worked because it was easy to choose principal axes **by symmetry**.

But **any** rigid body, however asymmetrical has 3 principal axes.

Thm: Given inertia tensor $\underline{\underline{I}}$ for any 3 axes, $\exists$ another set of principal axes such that $\underline{\underline{I}}'$ is diagonal

$$\underline{\underline{I}}' = \begin{bmatrix} I_{xx}' & 0 & 0 \\ 0 & I_{yy}' & 0 \\ 0 & 0 & I_{zz}' \end{bmatrix}$$

$$\mathcal{O} \, \underline{\underline{I}} \, \mathcal{O}^{-1} = \underline{\underline{I}}'$$

$\hookrightarrow$ orthogonal transformation (rotation of coordinates)

Solve

<u>Eigenvalue eqn</u> to find princ. axes

$$\underline{\underline{I}}\,\vec{\omega} = \lambda\,\vec{\omega}$$

or $$(\underline{\underline{I}} - \lambda \mathbb{I})\,\vec{\omega} = 0$$

solution only if $\det(\underline{\underline{I}} - \lambda \mathbb{I}) = 0$

"characteristic equation" for matrix $\underline{\underline{I}}$

$\underline{\underline{I}}$ is 3×3 => cubic polynomial in λ

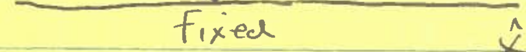
Find 3 possible eigenvalues λ_i $i = 1,2,3$

=> 3 eqns $\underline{\underline{I}}\,\vec{\omega}_i = \lambda_i\,\vec{\omega}_i$

2a Tennis Racket Thm

3 eigenvectors $\vec{\omega}_i$ determine 3 axes

Precession of top



Principal axes $\hat{e}_3$, $\hat{e}_1, \hat{e}_2 \perp \hat{e}_3$

$\vec{\omega} = \vec{\omega}(t)$ $\vec{L} = \lambda_3(\omega\,\hat{e}_3)$ initially as shown

$\vec{\tau} = \vec{R}\times M\vec{g} = \dot{\vec{L}}$ into page at $t=0$

(approx. for large ω_3) and small nonzero ω_2, ω_1 arise $\ll \omega$

=> top spins with $\sim\vec{\omega} \parallel \hat{e}_3$, but <u>direction</u> of $\hat{e}_3$ changes

Theorem – not going to prove

"Tennis Racket theorem"
or "Intermediate axis theorem"

Suppose $I_1 < I_2 < I_3$

rotation about I_2 is <u>unstable</u>

See video 4222 links page

Physics Girl + Rodney Mullen

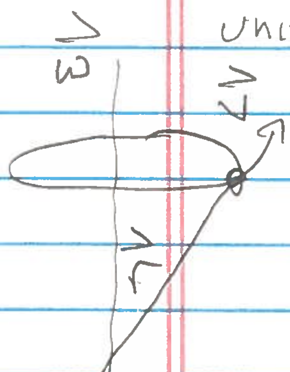
$$\vec{R} = R\hat{e}_3 \qquad g = -g\hat{z}$$

$$\lambda_3 \omega \dot{\hat{e}}_3 = \vec{R} \times M\vec{g}$$

$$\dot{\hat{e}}_3 = \frac{RMg}{\lambda_3 \omega}\, \hat{e}_3 \times (-\hat{z}) = \frac{RMg}{\lambda_3 \omega}\, \hat{z} \times \hat{e}_3$$

$$\equiv \vec{\Omega} \times \hat{e}_3 \quad \text{with } \vec{\Omega} \equiv \frac{MgR}{\lambda_3 \omega}\, \hat{z}$$

$\vec{\Omega} \parallel \hat{z}$ $\quad \dot{\hat{e}}_3 = \vec{\Omega} \times \hat{e}_3$ means axis $\hat{e}_3$ uniformly rotating around $\hat{z}$

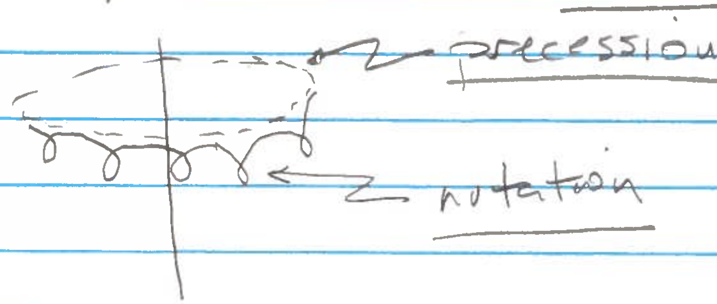


(recall $\dot{\vec{r}} = \vec{\omega} \times \vec{r}$)

So $\Omega = \dfrac{MgR}{\lambda_3 \omega}$ is ang. vel. of precession

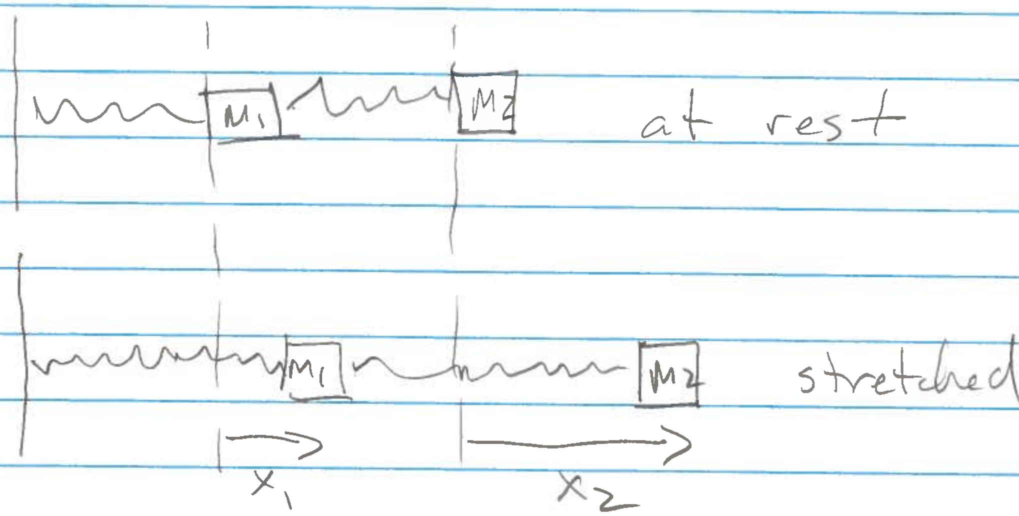
What about if torque is not weak, or ω is slow? Need full eqns. of motion, see Sec. 10.10 in Taylor.

Additional phenomenon: nutation



Ch. 11 Coupled oscillations

Old problem we solved w/ Lagrangian:



Newton:

F on 1: $-k_1 x_1 + k_2(x_2 - x_1)$

$= -(k_1+k_2)x_1 + k_2 x_2$

F on 2: $-k_2(x_2 - x_1)$

$\Rightarrow$ $m_1 \ddot{x}_1 = -(k_1+k_2)x_1 + k_2 x_2$

$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$

write as

$$\underline{M}\,\ddot{\vec{x}} = -\underline{K}\,\vec{x} \qquad *$$

$$\underline{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

"dynamical matrix"

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \underline{K} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Guess solns. to eqns. of motion:

$$x_1 = \alpha_1 \cos(\omega t + \delta_1)$$

$$x_2 = \alpha_2 \cos(\omega t + \delta_2)$$

Note the guess has same ω for both m's!

could have picked $\left.\begin{matrix} y_1 = \alpha_1 \sin(\omega t + \delta_1) \\ y_2 = \alpha_2 \sin(\omega t + \delta_2) \end{matrix}\right\}$ also

Combine into complex soln

$$z_1 = a_1 e^{i\omega t}$$

$$z_2 = a_2 e^{i\omega t}$$

$$a_1 = \alpha_1 e^{-i\delta_1} \qquad a_2 = \alpha_2 e^{-i\delta_2}$$

(verify $\mathrm{Re}\, z_1 = x_1$, $\mathrm{Im}\, z_1 = y_1$, etc.)

$$\vec{z} \equiv \vec{a}\, e^{i\omega t}, \qquad \vec{a} = \begin{pmatrix} \alpha_1 e^{-i\delta_1} \\ \alpha_2 e^{-i\delta_2} \end{pmatrix} \quad \underline{\text{complex}}$$