10/3/22

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Announcements

Exam 1/3 short answer 4/8
(see summaries at back of chapters,
PH lecture notes online)

+ 2 "word problems" 4 pts. each

HW 4 posted next wk, due 10/21

After test begin reading Ch. 9
Noninertial Frames

Term paper prospectus due 10/12

Test review

Examples:

1. Two particles orbit around each other with period τ. If they are suddenly stopped, show gravity takes $t = \frac{\tau}{4\sqrt{2}}$ to bring them together. Assume circles.

Soln: Eqn of motion before stop

$$\mu \ddot{r} = -\frac{\partial U}{\partial r} + \frac{l^2}{\mu r^3} = -\frac{G m_1 m_2}{r^2} + \underbrace{\frac{l^2}{\mu r^3}}_{}^{\text{cf}}$$

When planets are stopped $l \to 0$ so cf term vanishes. How do we relate distance a to period τ? Kepler's 3rd law!

stopping dist, $r = a$

$$\tau^2 = \frac{4\pi^2 \mu a^3}{G m_1 m_2} \Rightarrow a = \left(\frac{G m_1 m_2 \tau^2}{4\pi^2 \mu}\right)^{1/3}$$

So how long for m_1, m_2 to come together from this distance if they start at rest?

$\dot{r} = \dot{r}(r)$

$$\text{Note } \mu \ddot{r} = \mu \frac{d\dot{r}}{dr}\dot{r} = -\frac{G m_1 m_2}{r^2}$$

$\frac{d}{dt} = \frac{dr}{dt}\frac{d}{dr}$

$$\mu \int_0^{\dot{r}} \dot{r}' d\dot{r}' = -G m_1 m_2 \int_a^r \frac{dr'}{r'^2}$$

$$= \frac{1}{2}\mu \dot{r}^2 = G m_1 m_2 \left(1/r - 1/a\right)$$

$$\dot{r} = \left(\frac{2 G m_1 m_2}{\mu}\right)^{1/2} \left(1/r - 1/a\right)^{1/2}$$

$$\left(\frac{\mu}{2GM_1M_2}\right)^{1/2} \overbrace{\int_a^0 \frac{dr}{\sqrt{1/r - 1/a}}}^{\frac{\pi}{2}a^{3/2}} = \int_0^t dt' = t \quad \text{to fall together}$$

integral can be done w/ trig substitution

$$t = \frac{\pi}{2} a^{3/2} \left(\frac{\mu}{2GM_1M_2}\right)^{1/2}$$

From Kepler, $a^{3/2} = \left(\frac{GM_1M_2}{4\pi^2\mu}\right)^{1/2}$

$$\Rightarrow t = \frac{\cancel{\pi}}{2}\frac{1}{(2\cancel{\pi})}\frac{1}{\sqrt{2}}\tau = \frac{\tau}{4\sqrt{2}}$$

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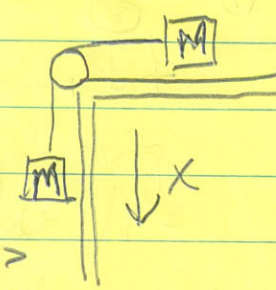
(2) Blocks of mass M arranged as on a frictionless table.



a) Lagrangian if string has no mass?

$$T = M\dot{x}^2 \qquad U = -Mgx \qquad \text{(see diag.)}$$

$$\mathcal{L} = T - U = M\dot{x}^2 + Mgx$$

E-L eqns. $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 2M\ddot{x} = Mg = \frac{\partial \mathcal{L}}{\partial x}$

$$\ddot{x} = \frac{g}{2} \checkmark \qquad x = \frac{1}{4}gt^2 \checkmark$$

b) Suppose string has mass m, length ℓ

Mass of <u>overhanging</u> string is $m\dfrac{x}{\ell}$

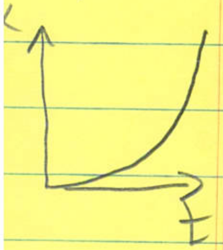
$$\boxed{\mathcal{L} = M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + Mgx + \frac{x^2}{2\ell}mg}$$

where last term is P.E. of c.m. of overhanging spring $\left(m\dfrac{x}{\ell}\right)g\left(\dfrac{x}{2}\right)$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \boxed{(2M+m)\ddot{x} = Mg + \frac{x}{\ell}mg} = \frac{\partial \mathcal{L}}{\partial x}$$

solve w/ lin. comb. $e^{\alpha t}$, $e^{-\alpha t}$

(Soln. $x(t) = \dfrac{\ell M}{m}\left(\cosh \alpha t - 1\right)$; $\alpha = \left(\dfrac{mg}{(2M+m)\ell}\right)^{1/2}$
for initial conditions $x(0) = \dot{x}(0) = 0$)



c) Find Hamiltonian (formally)

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = 2M\dot{x} + m\dot{x} \checkmark$$

$$H = p\dot{x} - \mathcal{L} = (2M+m)\dot{x}^2 - \mathcal{L}$$

$$= \left(M + \frac{m}{2}\right)\dot{x}^2 - Mgx - \frac{x^2}{2\ell}mg$$

$$= \frac{1}{2(2M+m)}p^2 - Mgx - \frac{x^2}{2\ell}mg$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{(2M+m)} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial x} = Mg + \frac{xmg}{\ell}$$

$$\Rightarrow \ddot{x} = \frac{\dot{P}}{(2M+m)} = \frac{Mg + xmg/l}{(2M+m)}$$ same as b)

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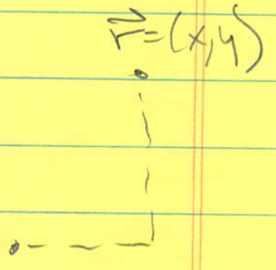
(3) Particle moves in 2D subject to force $\vec{F} = -kx\,\hat{x} + K\hat{y}$

Write down H + find motion.



$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad U = -\int_0^{\vec{r}} \vec{F}\cdot d\vec{r}$$

$$= -\int_0^{\vec{r}} (-kx\,dx) + (K\,dy)$$

$$= \tfrac{1}{2}kx^2 - Ky$$

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = m\dot{\vec{r}}$$

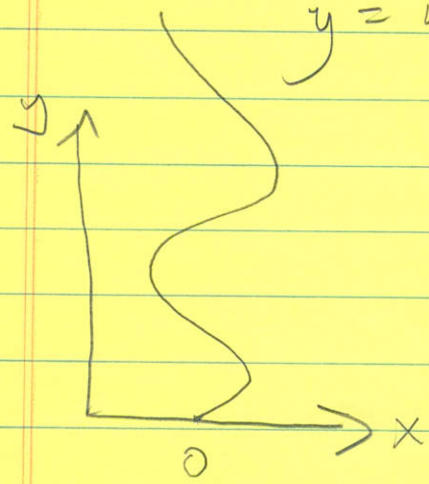
$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \tfrac{1}{2}kx^2 - Ky$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}; \quad \dot{y} = \frac{p_y}{m} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -kx$$

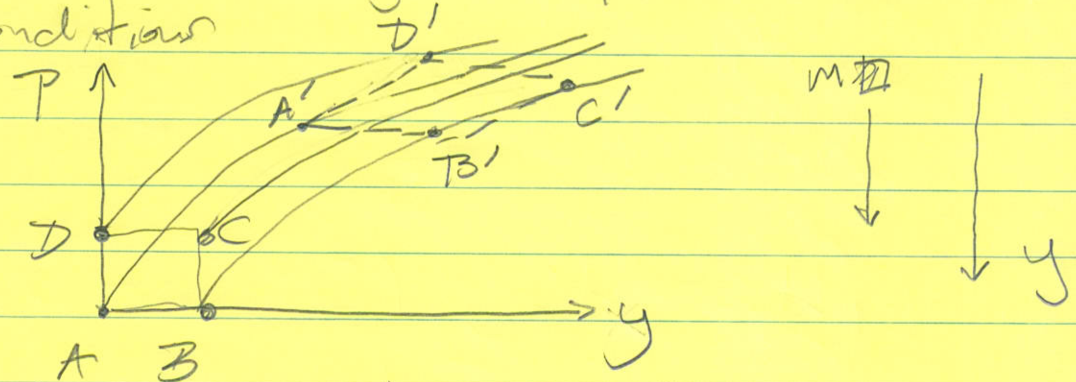
$$\dot{p}_y = K$$

$$m\ddot{x} = -kx \quad m\ddot{y} = K$$

So motion is combined SHO in x direction, $x = A\cos(\omega t + \delta)$
$$\omega = \sqrt{k/m}$$
plus free acceleration in y direction
$$y = y_0 + v_{0y}t + \frac{1}{2}\frac{K}{m}t^2$$



(4) Consider trajectories in phase space for a falling body with different initial conditions



Show A'B'C'D' is //ogram if ABCD is rectangle, with same area.

$$y = y_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2 \; ; \quad p = p_0 + mgt$$

	time 0			time t	
	y	p		y	p
A	0	0	A'	$\frac{1}{2}gt^2$	mgt
B	y_0	0	B'	$y_0 + \frac{1}{2}gt^2$	mgt
C	y_0	p_0	C'	$y_0 + \frac{p_0 t}{m} + \frac{1}{2}gt^2$	$p_0 + mgt$
D	0	p_0	D'	$\frac{p_0}{m}t + \frac{1}{2}gt^2$	$p_0 + mgt$

D, C & D', C' have same p-coord, p_0
A, B + A', B' " "

length of A'B' and C'D' is y_0
" " AB CD

$\Rightarrow$ both are $\parallel$ograms

$$\left.\begin{array}{l} \text{Both have bases} = y_0 \\ \text{heights} = p_0 \end{array}\right\} \Rightarrow \text{equal areas}$$