11/2/22

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Announcements

Quizzes 11/4 (moments of inertia)
11/7 (normal modes of osc.)

Last time

- principal axes of inertia

(diagonalize inertia tensor with orthogonal transformation

$$\underline{\underline{I}}' = \underline{\underline{O}}\,\underline{\underline{I}}\,\underline{\underline{O}}^{-1} = \begin{bmatrix} I_{xx}' & & 0 \\ & I_{yy}' & \\ 0 & & I_{zz}' \end{bmatrix}$$

Solve $\underline{\underline{I}}\,\vec{\omega} = \lambda\,\vec{\omega}$

Evalues $\det(\underline{\underline{I}} - \lambda \mathbb{1}) = 0$

if $\underline{\underline{I}}$ is 3×3 $\Rightarrow$ 3 evalues corresponding to 3 E-vectors $\vec{\omega}$ determine principal axes

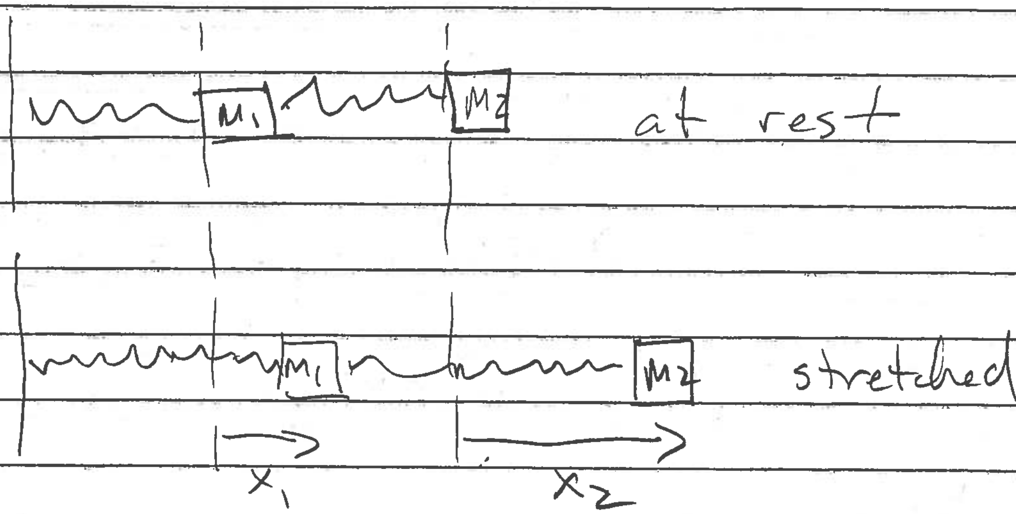
- Precession of top

use $\vec{\tau} = \dot{\vec{L}}$ to find prec. freq. $\Omega = \dfrac{mgR}{\lambda_3 \omega}$

λ_3 = moment of inertia

Ch. 11 Coupled oscillations

Old problem we solved w/ Lagrangian:



Newton:

F on 1 : $-k_1 x_1 + k_2 (x_2 - x_1)$

$$= -(k_1 + k_2) x_1 + k_2 x_2$$

F on 2 : $-k_2 (x_2 - x_1)$

$$\Rightarrow \quad m_1 \ddot{x}_1 = -(k_1 + k_2) x_1 + k_2 x_2$$
$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1)$$

write as

$$\underline{M} \ddot{\vec{x}} = - \underline{K} \vec{x} \qquad (*)$$

$$\underline{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

(OB)

"dynamical matrix"

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \underline{K} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Guess solns. to eqns. of motion:

$$x_1 = \alpha_1 \cos(\omega t + \delta_1)$$

$$x_2 = \alpha_2 \cos(\omega t + \delta_2)$$

Note the guess has same ω for both m's!

could have picked $\left.\begin{array}{l} y_1 = \alpha_1 \sin(\omega t + \delta_1) \\ y_2 = \alpha_2 \sin(\omega t + \delta_2) \end{array}\right\}$ also

Combine into complex soln

$$z_1 = a_1 e^{i\omega t}$$

$$z_2 = a_2 e^{i\omega t}$$

$a_1 = \alpha_1 e^{-i\delta_1} \qquad a_2 = \alpha_2 e^{-i\delta_2}$ also complex

(verify $\mathrm{Re}\, z_1 = x_1$, $\mathrm{Im}\, z_1 = y_1$, etc.)

$\vec{z} \equiv \vec{a}\, e^{i\omega t}$, $\quad \vec{a} = \begin{pmatrix} \alpha_1 e^{-i\delta_1} \\ \alpha_2 e^{-i\delta_2} \end{pmatrix}$ complex substitute $\vec{\xi} = \vec{a} e^{i\omega t}$ into ⊛

$$-\omega^2 M a \cancel{e^{i\omega t}} = -K a \cancel{e^{i\omega t}}$$

$$\boxed{(\underline{K} - \omega^2 \underline{M})\vec{a} = 0} \qquad \#$$

eigenval eqn $\Rightarrow$ $\det(\underline{K} - \omega^2 \underline{M}) = 0$

$$\boxed{\begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 M_2 \end{vmatrix} = 0}$$

$$(k_2 - \omega^2 m_2)(k_1 + k_2 - \omega^2 m_1) - k_2^2 = 0$$

$$0 = k_2 k_1 - k_2 \omega^2 m_1 - (k_1 + k_2)\omega^2 m_2 + \omega^4 m_1 m_2$$

$$\omega^2 = \frac{m_1 k_2 + m_2(k_1 + k_2) \pm \sqrt{(m_1 k_2 + m_2(k_1 + k))^2 - 4 k_1 k_2 m_1 m_2}}{2 m_1 m_2}$$

Does this make any sense?
Check limit $m_1 = m_2 = m$, $k_1 = k_2 = k$

$$\omega^2 = \frac{3km \pm \sqrt{5k^2m^2}}{2m^2} = \frac{3 \pm \sqrt{5}}{2} \frac{k}{m}$$

$\underline{K} - \omega^2 \underline{M} =$

As with any eigenvalue problem, we plug ω^2 back into # to get sol'n

Eigenvalue 1

$$\omega_1^2 = \frac{3-\sqrt{5}}{2}\ \frac{k}{m}$$

$$k_1 + k_2 - \omega^2 m_1 \longrightarrow 2k - \left(\frac{3-\sqrt{5}}{2}\right)k = \left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)k$$

$$\underline{K} - \omega^2 \underline{M} = k\begin{bmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix}$$

Since $(\underline{K} - \omega^2 \underline{M})\vec{a} = 0$, this means

$$\left.\begin{aligned} \frac{1+\sqrt{5}}{2}a_1 - a_2 &= 0 \\ \frac{-1+\sqrt{5}}{2}a_2 - a_1 &= 0 \end{aligned}\right\} \text{ same eqn}$$

only learn $\dfrac{a_1}{a_2} = \left(\dfrac{-1+\sqrt{5}}{2}\right)$

A real

So $$\vec{z}(t) = \begin{bmatrix} A\left(\frac{-1+\sqrt{5}}{2}\right) \\ A \end{bmatrix} e^{i(\omega_1 t - \delta_1)}$$

Eigenvalue 2 $\quad \omega_2^2 = \dfrac{3+\sqrt{5}}{2}\sqrt{\dfrac{k}{m}}$

$$\underline{K} - \omega^2 \underline{M} = \frac{k}{m}\begin{bmatrix} \frac{1-\sqrt{5}}{2} & -k \\ -k & \frac{-1-\sqrt{5}}{2} \end{bmatrix}$$

$$\left.\begin{array}{l} \frac{1-\sqrt{5}}{2} a_1 - a_2 = 0 \\ -\frac{1-\sqrt{5}}{2} a_2 - a_1 = 0 \end{array}\right\} \text{ same eqn} \Rightarrow \frac{a_1}{a_2} = \frac{-1-\sqrt{5}}{2}$$

$$\vec{z}(t) = \begin{bmatrix} A\left(\frac{-1-\sqrt{5}}{2}\right) \\ A \end{bmatrix} e^{i(\omega_2 t - \delta_2)}$$

Two different eigenvectors correspond to two different <u>normal modes</u>

2 masses oscillate at <u>same frequency</u>, different however for each normal mode.

N.B. We have just said that if you get the system going in this mode, it will continue this way. How exactly to excite the mode may be tricky.

General soln: $A_1 \begin{bmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{bmatrix} e^{i(\omega_1 t - \delta_1)} + A_2 \begin{bmatrix} \frac{1\pm\sqrt{5}}{2} \\ 1 \end{bmatrix} e^{i(\omega_2 t - \delta_2)}$

<u>lin. comb. of normal modes</u>

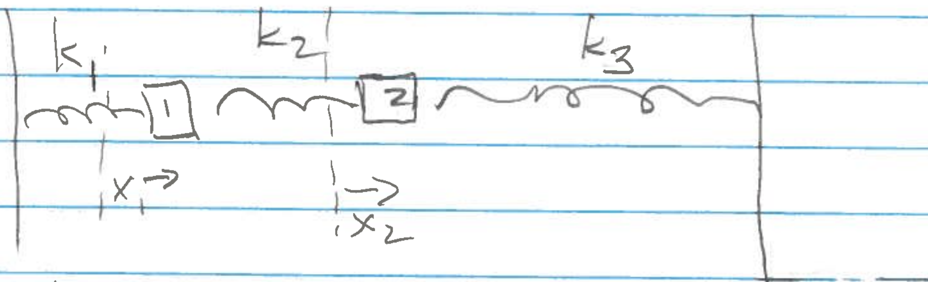
Re part

$$\vec{x} = A_1 \begin{bmatrix} 1 \\ \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 2 \\ \end{bmatrix} \cos(\omega_2 t - \delta_2)$$

Weakly coupled oscillators

See example in section 11.1-11.2 in book.

with l



+ assume $k_1 = k_2 = k$, $m_1 = m_2$, $k_2 << k$

In general $m\ddot{x}_1 = -(k+k_2)x_1 + k_2 x_2$

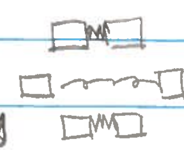
$$m\ddot{x}_2 = k_2 x_1 - (k_2 + k)x_2$$

$$\Rightarrow \underline{K} = \begin{bmatrix} k+k_2 & -k_2 \\ -k_2 & k_2+k \end{bmatrix}$$

$$\underline{K} - \omega^2 \underline{M} = \begin{bmatrix} k+k_2-m\omega^2 & -k_2 \\ -k_2 & k_2+k-m\omega^2 \end{bmatrix}$$

$$\det\left[\ \ \right] = (k - m\omega^2)(k + 2k_2 - m\omega^2)$$

$$\omega = \sqrt{\frac{k}{m}}, \quad \omega = \sqrt{\frac{k+2k_2}{m}}$$



2 carts oscillate back + forth together w/o compressing k_2

"breathing mode"