11/4/22

Announcements

Quiz today, also Monday (normal modes)
HW 6 due 11/14

Last time Normal modes

System of coupled oscillators described by set of linear eqns

$$\sum_j M_{ij}\ddot{x}_j = -\sum_j K_{ij} x_j \qquad i = 1, 2, 3, \ldots \ldots N \text{ oscillators}$$

(1) $$\underline{M}\,\ddot{\vec{x}} = -\underline{K}\,\vec{x}$$

search for normal modes - solns with one freq ω

(2) $$\Rightarrow \left(\underline{M}\omega^2 - \underline{K}\right)\vec{a} = 0$$

$\vec{a}$ – vector of amplitudes

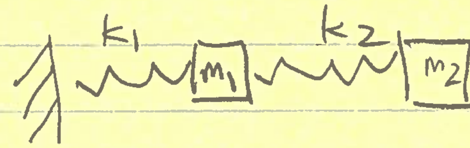
ω^2 is eigenvalue

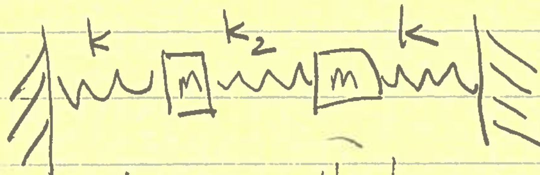
(3) $\Rightarrow$ solve $\det\left(\underline{M}\omega^2 - \underline{K}\right) = 0$

polynomial $\mathcal{O}(N)$ in ω^2

④ find Evectors (normal modes)

/ for each Evalue ω^2

— Solved 

— Started 

$k_2 \ll k$

weakly coupled oscillators

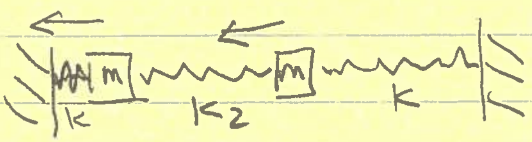
$$\underline{K} = \begin{bmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{bmatrix} \qquad \underline{m} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

exact

found $\omega^2 = \underset{1}{\frac{k}{m}}, \ \underset{2}{\frac{k+2k_2}{m}}$

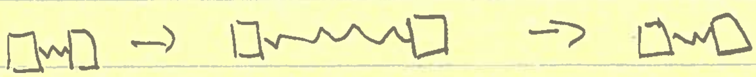
Guess modes:

① $\omega = \sqrt{\frac{k}{m}}$ 2 osc. in phase



k_2 never compressed!

② $\omega = \sqrt{\frac{k+2k_2}{m}}$ "breathing mode"



<u>Note</u> That since $k_2 \ll k$,

$$\omega_1 \approx \omega_2$$

so express in terms of sum + difference frequencies

$\omega_1 = \omega_0 - \varepsilon$

$\omega_2 = \omega_0 + \varepsilon$

$$\omega_0 \equiv \frac{\omega_1 + \omega_2}{2} \qquad \varepsilon \equiv \frac{\omega_2 - \omega_1}{2}$$

Find normal modes (eigenvectors)

$\omega_1^2 = \frac{k}{m}$

$$(\underline{K} - \omega_1^2 \underline{M}) = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

ω_1

$$\Rightarrow \underline{z}_1(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0 - \varepsilon)t}$$

$\omega_2^2 = \sqrt{\frac{k+2k}{m}}$

$$(\underline{K} - \omega_2^2 \underline{M}) = \begin{bmatrix} -k_2 & -k_2 \\ -k_2 & -k_2 \end{bmatrix}$$

ω_2

$$\Rightarrow \underline{z}_2(t) = C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0 + \varepsilon)t}$$

gen. soln $\quad z = z_1 + z_2$ $\left\{ \text{w/arbitrary complex consts. } C_1, C_2 \right.$

$$= \left[C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\varepsilon t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\varepsilon t} \right] e^{i\omega_0 t}$$

NB – this is where exponential notation comes in handy!

To see character of full soln,

take $C_1 = C_2 = A/2$ real

$$\Rightarrow \quad \vec{z}(t) = A \begin{bmatrix} \cos \varepsilon t \\ -i \sin \varepsilon t \end{bmatrix} e^{i\omega_0 t}$$

Still need $\vec{x}(t) = \mathrm{Re}\, \vec{z}(t)$

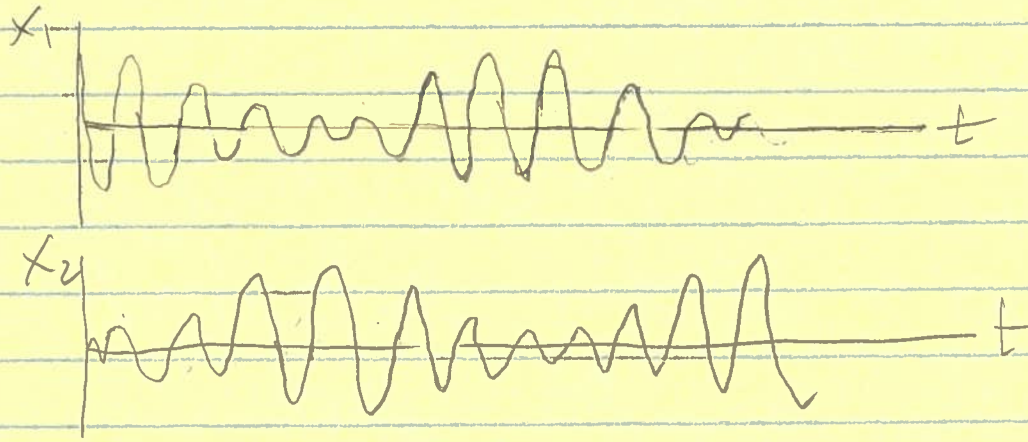
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} \cos \varepsilon t \cos \omega_0 t \\ \sin \varepsilon t \sin \omega_0 t \end{bmatrix}$$

Note initial conditions are implicitly assumed with Re C_1, C_2:

check: $\underline{x_1(0) = A \quad x_2(0) = 0}$ $\quad \underline{\dot{x}_1(0) = \dot{x}_2(0)} = 0$ $\left\{ \begin{array}{l} \text{Mass 1 pulled } A \text{ to right,} \\ \text{" " 2 left stationary,} \\ \text{both released from rest} \end{array} \right.$

recall: ω_0 is fast oscillation $\quad \varepsilon$ is slow

out of phase 90°



"Beats" as in sound waves Phys. I