11/7/22

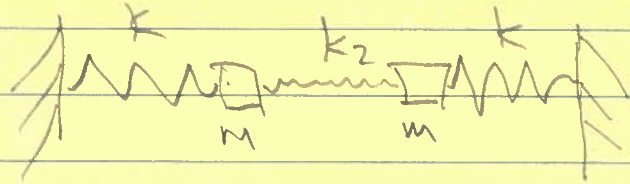
Announcements

HW 6 due 11/14
Tues ofc. hrs cancelled

Last time

Finding normal mode eigenvectors, constructing general soln

— 2 weakly coupled oscillators



$k_2 \ll k$

solved $\left(\underline{K} - \omega^2 \underline{M}\right)\vec{a} = 0$

Evalues — Evectors

① $\omega_1^2 = \frac{k}{m}$ $\quad \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in-phase

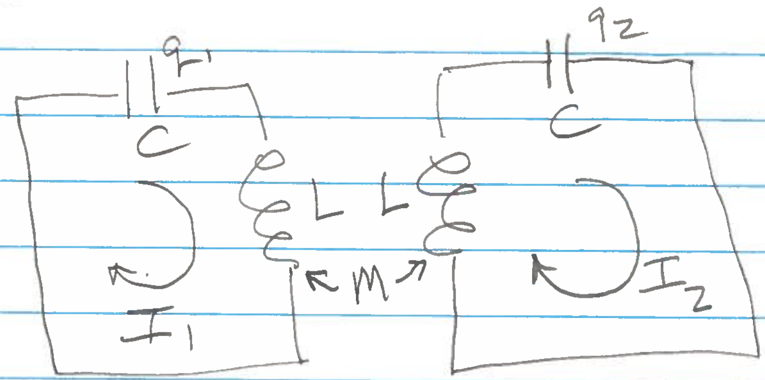
② $\omega_2^2 = \frac{k+2k_2}{m}$ $\quad \vec{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ out of phase

Gen'l soln. $C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0+\varepsilon)t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i(\omega_0-\varepsilon)t}$

where $\omega_1 = \omega_0 + \varepsilon$, $\omega_2 = \omega_0 - \varepsilon$, $\varepsilon \ll \omega_0$

For $C_1 = C_2$ found beats, out of phase by 90°

Example 2 coupled electrical circuits



Consider 2 identical LC circuits oriented to couple via their mutual inductance

Kirchoff:
$$L\dot{I}_1 + \frac{q_1}{C} + M\dot{I}_2 = 0$$
$$L\dot{I}_2 + \frac{q_2}{C} + M\dot{I}_1 = 0$$

Differentiate wrt time: $\frac{dq_\alpha}{dt} = I_\alpha$

L = self ind.
M = mutual ind

$$L\ddot{I}_1 + \frac{I_1}{C} + M\ddot{I}_2 = 0$$
$$L\ddot{I}_2 + \frac{I_2}{C} + M\ddot{I}_1 = 0$$

of form $\underline{L}\,\ddot{\vec{I}} = -\underline{K}\,\vec{I}$ $\qquad \vec{I} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$

$$\underline{L} = \begin{bmatrix} L & M \\ M & L \end{bmatrix} \qquad \underline{K} = \begin{bmatrix} 1/C & 0 \\ 0 & 1/C \end{bmatrix}$$

$$\underline{K} - \omega^2 \underline{L} = \begin{bmatrix} 1/C - \omega^2 L & -\omega^2 M \\ -\omega^2 M & 1/C - \omega^2 L \end{bmatrix}$$

Det (secular eqn.) is

$$\left(1/C - \omega^2 L\right)^2 - \omega^4 M^2 = 0$$

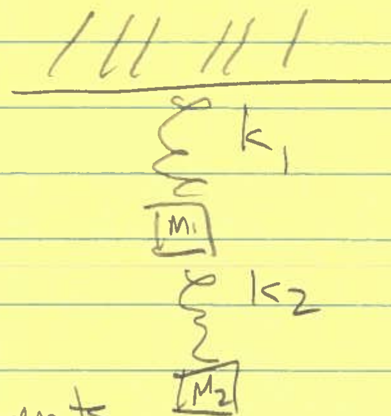
Solutions $\omega^2 = \dfrac{1}{LC \pm MC}$

Resonant ("normal mode") frequencies are below and above resonant frequency of isolated circuit $\omega^2 = 1/LC$

Note if far apart $M \ll L \Rightarrow$ beats

Q: why can you neglect equil. positions?

Examples:



1 Massless springs vertical - 2 masses
Find $\underline{M}$ + $\underline{K}$

x_1 and x_2 are displacements from unstretched lengths
x_{1eq} and x_{2eq} displacements at equil.

Coordinates $y_1 = x_1 - x_{1eq}$ $\quad y_2 = x_2 - x_{2eq}$

Downward
$$F_1 = m_1 g - k_1 x_1 + k_2(x_2 - x_1)$$
$$F_2 = m_2 g - k_2(x_2 - x_1)$$

Equilibrium:
$$0 = m_1 g - k_1 x_{1eq} + k_2(x_{2eq} - x_{1eq})$$
$$0 = m_2 g - k_2(x_{2eq} - x_{1eq})$$

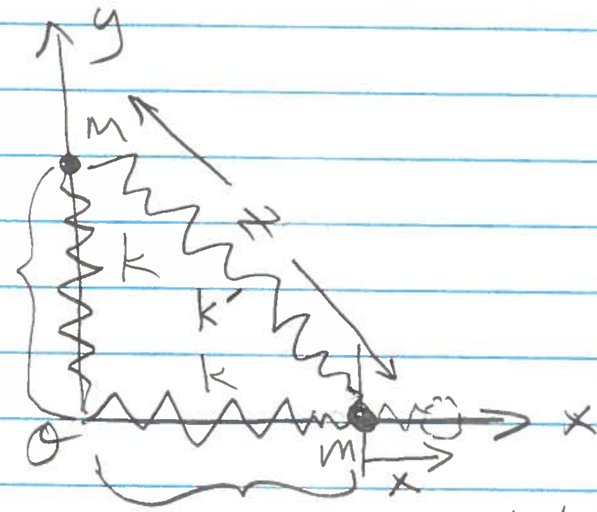
$$m_1 \ddot{y}_1 = F_1 = \cancel{\underline{m_1 g}} - k_1(y_1 + \cancel{\underline{x_{1eq}}}) + k_2(y_2 - y_1 + \cancel{\underline{x_{2eq}}} - \cancel{\underline{x_{1eq}}})$$
$$= -k_1 y + k_2(y_2 - y_1)$$

also $m_2 \ddot{y}_2 = -k_2(y_2 - y_1)$

$$\underline{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \underline{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \checkmark$$

Moral of story - set up in terms of <u>deviations</u> from equilibrium right away.

Coupled oscillators w/ Lagrangian method



2 equal masses connected to origin by spring k, also to each other by k'. Each can only move on x or y axis, respectively. k, k' chosen such that in equilibrium k, k' have their unstretched lengths $L, L, \sqrt{2}\,L$. Find normal modes.

L = unstretched

Soln:

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$U = \tfrac{1}{2} k\left(x^2 + y^2\right) + \tfrac{1}{2} k' z^2$$

$$z = \sqrt{(L+x)^2 + (L+y)^2} - \sqrt{2}\,L$$

expand for $x \ll L$, $y \ll L$

$$\simeq \sqrt{2L^2 + 2L(x+y)} - \sqrt{2}\,L$$

$$= \sqrt{2}\,L\left(\sqrt{1 + \frac{(x+y)}{L}} - 1\right) \simeq \sqrt{2}L\,\frac{(x+y)}{2L} = \frac{\sqrt{2}}{2}(x+y)$$

$$\Rightarrow U \simeq \tfrac{1}{2}k\left(x^2+y^2\right) + \tfrac{1}{4}k'(x+y)^2$$

$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} \Rightarrow$$

$$m\ddot{x} = -kx - \tfrac{1}{2}k'x - \tfrac{1}{2}k'y$$

similarly for y: $m\ddot{y} = -ky - \tfrac{1}{2}k'y - \tfrac{1}{2}k'x$

we need $\underline{K} = \begin{bmatrix} k+k'/2 & k'/2 \\ k'/2 & k+k'/2 \end{bmatrix}$

$\underline{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

$$\underline{K} - \omega^2 \underline{M} = \begin{bmatrix} k+k'/2 - m\omega^2 & k'/2 \\ k'/2 & k+k'/2 - m\omega^2 \end{bmatrix}$$

$$\det(\underline{K} - \omega^2\underline{M}) = \underbrace{(m\omega^2 - k)(m\omega^2 - k - k')}_{\text{factorizes}}$$

Eigenfrequencies:

$$\omega_1^2 = k/m \qquad \omega_2^2 = \frac{k+k'}{m}$$

E'val. ① $\quad \underline{K} - \omega^2\underline{M}\Big|_{\omega_1} = \begin{bmatrix} k'/2 & k'/2 \\ k'/2 & k'/2 \end{bmatrix}$

$\omega_1^2 = k/m \qquad = \frac{k'}{2}\,\mathbb{1}$

E-vector $\vec{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\vec{x}_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\omega_1 t}$

So as one m moves in +x direction, other m moves in −y direction! This is a "sloshing" back & forth. Note that length of k′ (2) is not changed to leading order in this motion ⟹ ω_1 does not depend on k′ !

E-vector 2 $\qquad \omega_2^{\,2} = \frac{k+k'}{m}$

$$\left(\underline{K} - \omega^2 \underline{M}\right)\Big|_{\omega_2} = \begin{bmatrix} -k'/2 & k'/2 \\ k'/2 & -k'/2 \end{bmatrix}$$

$$\vec{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_2 t}$$

Here two masses move together in +x, y directions simultaneously then back along −x, y axes together. k′ gets stretched in this one — a "breathing" mode.