11/9/22

Announcements

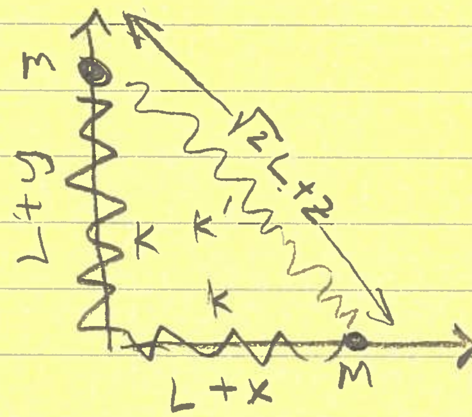
HW 6 due 11/14

No class Friday - Veteran's Day

PH travel - zoom classes 11/18, 11/21

Last time

- LC coupled circuits
- vertical coupled springs
- 3 springs



Both springs k:
Equil. length L

Spring k'
Equil. length $\sqrt{2}\,L$

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right)$$

$$U = \frac{1}{2} k \left(x^2 + y^2\right) + \frac{1}{2} k' z^2$$

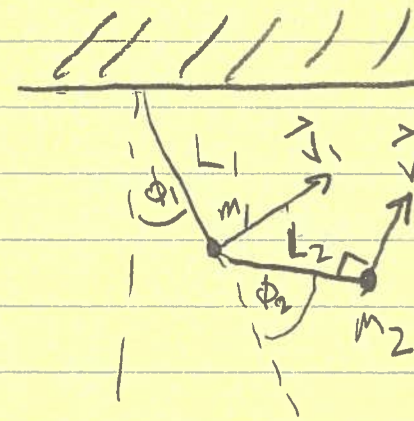
$$z = \sqrt{(L+x)^2 + (L+y)^2} - \sqrt{2}\,L$$

$$\simeq \frac{\sqrt{2}}{2}(x+y)$$

$$\Rightarrow U \simeq \frac{1}{2} k \left(x^2+y^2\right) + \frac{1}{4} k' (x+y)^2$$

See discussion pp. 4-5 in 11/7 notes for continuation.

Ex 4 Double pendulum



Lagrangian: $T - U$

$T = \frac{1}{2} m_1 L_1^2 \dot{\phi}_1^2$ OK

$\vec{v}'$ is velocity of m_2 <u>relative</u> to $\vec{v}_1$

$$\vec{v}_2 = \vec{v}_1 + \vec{v}'$$

$$|\vec{v}_1| = L_1 \dot{\phi}_1 \qquad \vec{v}' = L_2 \dot{\phi}_2$$

$$|\vec{v}_2|^2 = |v_1|^2 + |\vec{v}'|^2 + 2 \vec{v}_1 \cdot \vec{v}'$$

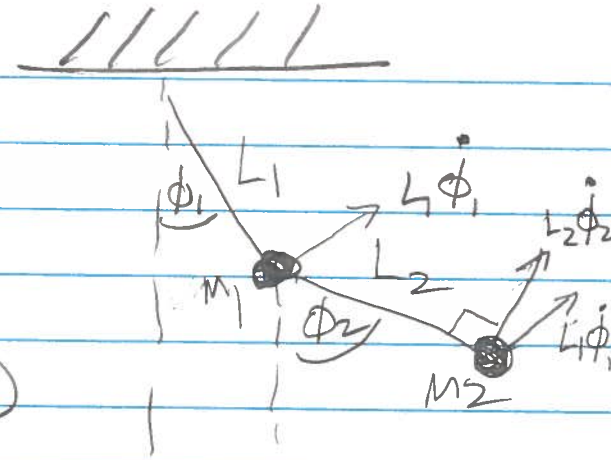
$$= L_1^2 \dot{\phi}_1^2 + L_2^2 \dot{\phi}_2^2 + 2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_2 - \phi_1)$$

$\Rightarrow$

$$T = \frac{1}{2} m_1 L_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 \left(L_1^2 \dot{\phi}_1^2 + L_2^2 \dot{\phi}_2^2 + 2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_2 - \phi_1) \right)$$

$$U_{tot} = (m_1 + m_2) g L_1 (1 - \cos\phi_1) + m_2 g L_2 (1 - \cos\phi_2)$$

Double pendulum



PE easy:

$$U_1 = m_1 g L_1 (1-\cos\phi_1)$$

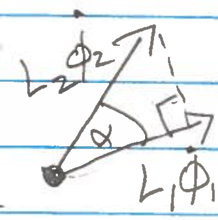
$$U_2 = m_2 g L_1 (1-\cos\phi_1) + m_2 g L_2 (1-\cos\phi_2)$$

$$U_{tot} = (m_1+m_2) g L_1 (1-\cos\phi_1) + m_2 g L_2 (1-\cos\phi_2)$$

$$T_1 = \tfrac{1}{2} m_1 L_1^2 \dot\phi_1^2 \quad \text{easy}$$

KE subtle:

velocity of m_2



$\alpha = \phi_2 - \phi_1$

$$|\vec{v}_2|^2 = |\vec{v} + \vec{v}'|^2 = v^2 + v'^2 + v\,v' \cos\alpha$$

$$= L_1^2\dot\phi_1^2 + L_2^2\dot\phi_2^2 + 2L_1L_2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)$$

11/3 → Exact Lagrange eqns. not exactly soluble (as for simple pendulum)

So assume small oscillations $\phi_1, \phi_2 \ll 1$

$$\dot\phi_1\dot\phi_2 \cos(\phi_1-\phi_2) \simeq \dot\phi_1\dot\phi_2 \quad \mathcal{O}(\text{small}^2)$$

$$\Rightarrow T = T_1 + T_2 \simeq \tfrac{1}{2} m_1 L_1^2 \dot\phi_1^2 + \tfrac{1}{2} m_2 L_1^2 \dot\phi_1^2 + \tfrac{1}{2} m_2 L_2^2 \dot\phi_2^2 + m_2 L_1 L_2 \dot\phi_1 \dot\phi_2$$

Approximate U up to order $(\text{small})^2$:

$$\cos\phi_\alpha = 1 - \phi_\alpha^2/2$$

$$U \simeq \frac{1}{2}(m_1+m_2)gL_1\phi_1^2 + \frac{1}{2}m_2 g L_2 \phi_2^2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\phi_1} = (m_1+m_2)L_1^2\ddot\phi_1 + m_2 L_1 L_2 \ddot\phi_2$$

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -(m_1+m_2)gL_1\phi_1$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\phi_2} = m_2 L_2^2\ddot\phi_2 + m_2 L_1 L_2 \ddot\phi_1$$

$$\frac{\partial \mathcal{L}}{\partial \phi_2} = -m_2 g L_2 \phi_2$$

$$\underline{M}\,\ddot{\vec\phi} = -\underline{K}\,\vec\phi$$

$$\underline{M} = \begin{bmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix}$$

$$\underline{K} = \begin{bmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix}$$

This is setup for full normal mode problem. Simplify a bit further so we can have some intuition for solution.

$m_1 = m_2 = m \qquad L_1 = L_2 = L$

$$\Rightarrow \quad \underline{M} = mL^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \underline{K} = mL^2 \begin{bmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix}$$

$$\omega_0 = \sqrt{g/L}$$

$$0 = \det\left(\underline{M}\,\omega^2 - \underline{K}\right) = \det(mL^2)\begin{pmatrix} 2(\omega^2 - \omega_0^2) & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix}$$

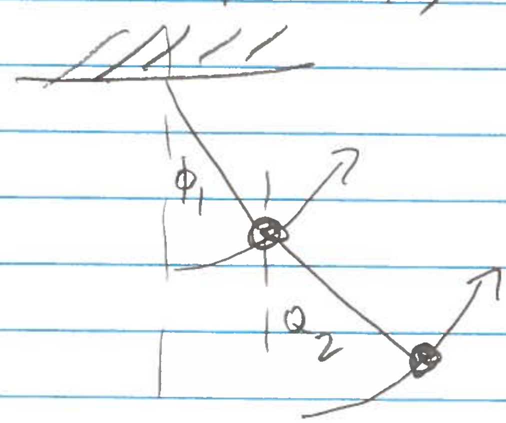
$$0 = 2(\omega^2 - \omega_0^2)^2 - \omega^4 = \omega^4 + 2\omega_0^4 - 4\omega_0^2\omega^2$$

$$= \left[\omega^2 - (2+\sqrt{2})\omega_0^2\right]\left[\omega^2 - (2-\sqrt{2})\omega_0^2\right]$$

$$\omega_{1,2}^2 = (2 \mp \sqrt{2})\omega_0$$

substitute, find modes

1st mode $\quad \vec{\phi} = A_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \cos(\omega_1 t - \delta_1)$



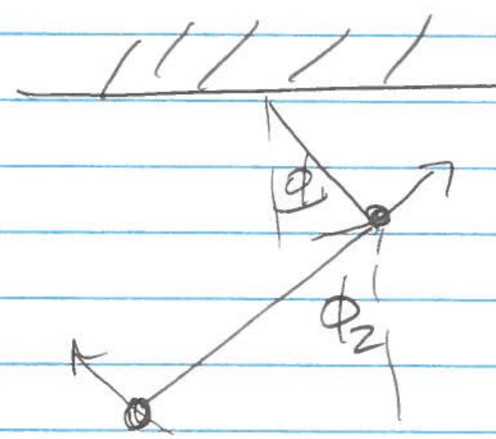
in phase!

$$\phi_1 = \sqrt{2}\,\phi_1$$

2nd mode ω_2

$$\vec{\phi} = A_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \cos(\omega_2 t - \delta)$$

ϕ_2 π out of phase with ϕ_1
amplitude still $\sqrt{2}$ bigger



Remember these solns only for $\phi_1, \phi_2 \ll 1$
so pictures are an exaggeration.

Full soln needs to be solved numerically, leads to chaos ⟶
see lobby demo!