12/2/22

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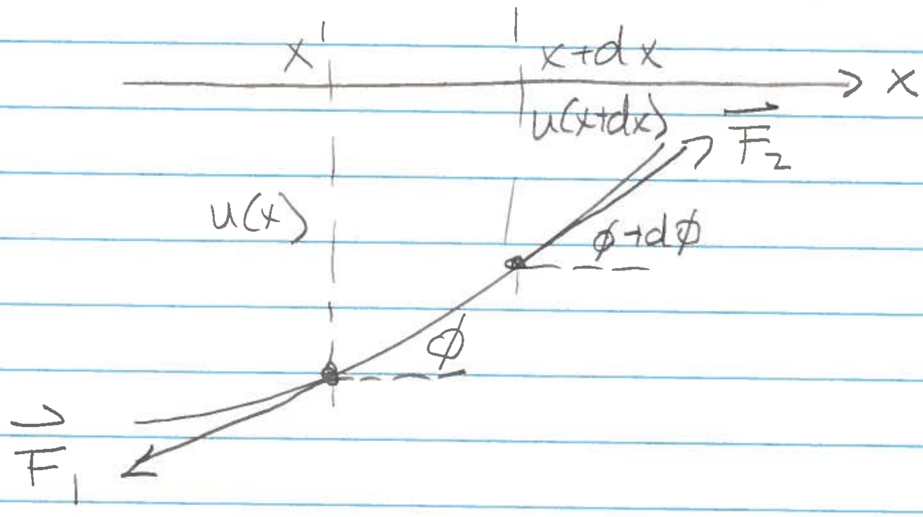
Announcements

- Opt. term papers due today
- HW 8 due 12/7
- Lectures today, Monday: waves
  Wednesday: review
  $\Rightarrow$ no relativity ☹
- Final 12/14 2/5 T1 2/5 T2 1/5 Ch. 11
- Course evals ?

Last time -

Continuum mechanics - wave eqn

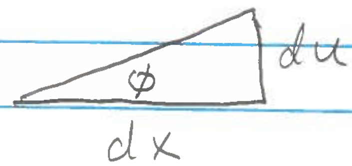


- assume displacements u small
  $\Rightarrow$ ϕ small

found $d\vec{F}^{tot} \cong T\, d\phi\, \hat{y}$
restoring force in $\hat{y}$ direction only
(in this approx.)

Now note that the slope of the string function is

$$\frac{\partial u}{\partial x} = \tan\phi \simeq \phi$$



$$\Rightarrow F_y^{tot} = T\,d\phi = T\frac{\partial \phi}{\partial x}dx$$

$$= T\frac{\partial^2 u}{\partial x^2}dx$$

$$\Rightarrow m a_y = m\frac{\partial^2 u}{\partial t^2} = \mu\frac{\partial^2 u}{\partial t^2}dx$$

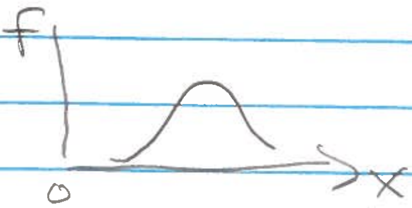
$\mu \rightarrow$ linear mass density

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}} \qquad \boxed{c^2 = T/\mu}$$

Particular solutions $u(x,t) = f(x \pm ct)$ where f is any smooth function

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x \pm ct) = c^2 f''(x \pm ct)$$

Check $f(x) =$



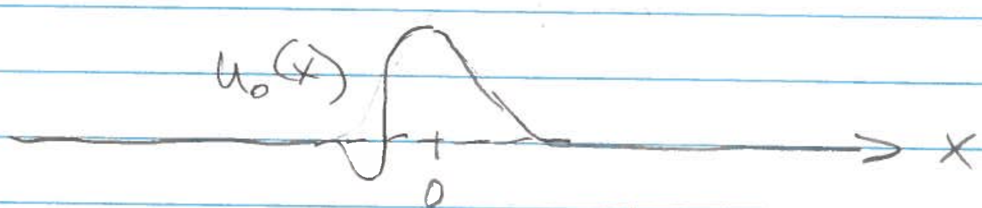
$f(x \mp ct)$ is wave traveling to right / left

General soln $u(x,t) = f(x+ct) + g(x-ct)$

could be different fct

Example: triangular wave

Pluck string in some shape at $t=0$



(Note strictly speaking, book's triangle pulse is not allowed, since $\partial^2 u/\partial x^2$ not defined at $x=0$, but it doesn't matter, here any $u_0(x)$ is ok)

Need to specify initial velocity as well

1) $u(x, t=0) = f(x) + g(x) = u_0(x)$
2) $u'(x, t=0) = -cf'(x) + cg'(x) = 0$

$$g(x) \overset{①}{=} u_0(x) - f(x)$$
$$g'(x) = u_0'(x) - f'(x) \overset{②}{=} f'(x)$$

$$\Rightarrow \quad 2f'(x) = u_0'(x)$$

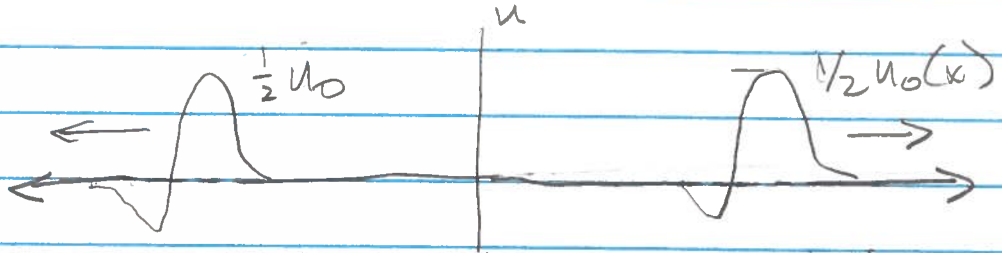
$$f'(x) = \tfrac{1}{2}u_0'(x) = g'(x)$$
$$f(x) = \tfrac{1}{2}u_0(x) + \text{const}$$
$$g(x) = \tfrac{1}{2}u_0(x) + \text{const}$$

1) $\Rightarrow$ const $= 0$

reconstruct full t-dependent soln.

$$\Rightarrow \quad u(x,t) = \tfrac{1}{2}u_0(x-ct) + \tfrac{1}{2}u_0(x+ct)$$

two pulses same shape moving apart at speed c



later time snapshot $t>0$

Normal modes

One soln. we often study for its simplicity is a traveling sinusoidal wave

argument is dimensionless!

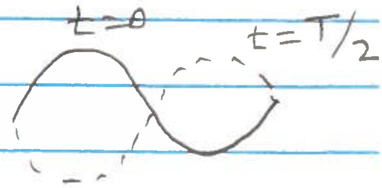
take
$$f = \sin(kx-\omega t) \qquad k = 2\pi/\lambda$$
$$g = \sin(kx+\omega t) \qquad \omega = 2\pi F$$
$$F\lambda = c$$

$\sin\left(\frac{2\pi}{\lambda}(x-ct)\right)$

Sum of wave traveling to left + right, amplitude 1:

$$\sin(kx-\omega t) + \sin(kx+\omega t)$$
$$= \sin kx \cdot \cos\omega t + \cancel{\cos kx \sin\omega t}$$
$$+ \sin x \cos ct - \cancel{\cos kx \sin\omega t}$$

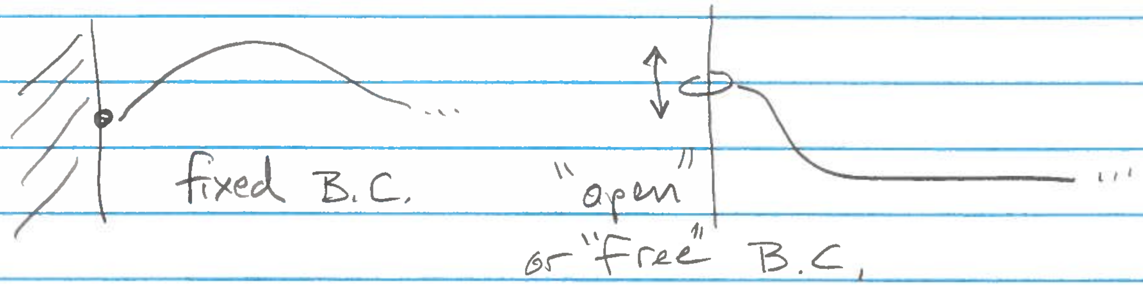
$$= 2\sin kx\cos\omega t$$



Standing wave!

Boundary conditions + normal modes

Consider finite string. 2 simple types of boundary conditions on the diffeq.:



Consider 1st string length L fixed at both ends: $u(0,t) = u(L,t) = 0$

Look for normal modes as before:

assume $u(x,t) = X(x)\cos(\omega t - \delta)$
i.e. one frequency only

Substitute in wave eqn:

$$-\omega^2 X \cancel{\cos(\omega t - \delta)} = c^2 \frac{d^2X}{dx^2} \cancel{\cos(\omega t - \delta)}$$

$$\frac{d^2X}{dx^2} = -k^2 X \qquad k = \omega/c$$

gen'l solution $a\cos kx + b\sin kx$

B.C. $x=0$ $\quad a = 0 \quad$ $x=L$ $\begin{cases} b = 0 & \text{trivial} \\ \sin kL = 0 \end{cases}$

To satisfy B.C., we need $k = k_n = \frac{\pi n}{L}$, $n = 1, 2, \ldots$

- Note there are an infinite number of solutions, in contrast to discrete mechanical system, where we had N degrees of freedom.

- The frequencies $\omega_n = \frac{\pi n}{L} c$ are the standing wave / normal mode frequencies

$n = 1$ is "fundamental"
$n = 2$ "1st harmonic" or "1st overtone"
3 2nd " " 2nd " "

wavelength of mode n is $\lambda_n = \frac{c}{f_n} = \frac{2\pi c}{\omega_n} = \frac{2L}{n}$

$\lambda_1 = 2L$ $\lambda_2 = L$ $\lambda_3 = \frac{2L}{3}$.

Any standing wave is a linear combination of these normal modes. In general, when you pluck a guitar string, you excite all of them, to varying degrees.

To see this, examine normal mode soln.

$$u(x,t) = \sin k_n x \left(A_n \cos(\omega_n t - \delta_n) \right)$$
$$= \sin k_n x \left(B_n \cos \omega_n t + C_n \sin \omega_n t \right)$$