8/26/22

1. Review of Lagrangian dynamics

A. Calculus of variations (Taylor ch. 6)

Euler-Lagrange Eq.:

$y(x_1), y(x_2)$ fixed

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\,dx$$

$\delta S = 0$ is stationary when $y(x)$ satisfies

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$$

B. Lagrange's eqns.

follow from Hamilton's principle

$$S \equiv \int_{t_1}^{t_2} \mathcal{L}\,dt$$

is stationary along true path $(x(t), y(t), z(t))$
i.e. $\delta S = 0$

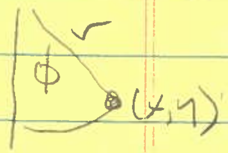
$$\Rightarrow \quad \boxed{\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}} \qquad i = 1, \ldots N$$

conservative holonomic systems

where q_i are generalized coordinates

$$\mathcal{L} = \mathcal{L}(q_1 \ldots q_N; \dot{q}_1 \ldots \dot{q}_N)$$

remarks



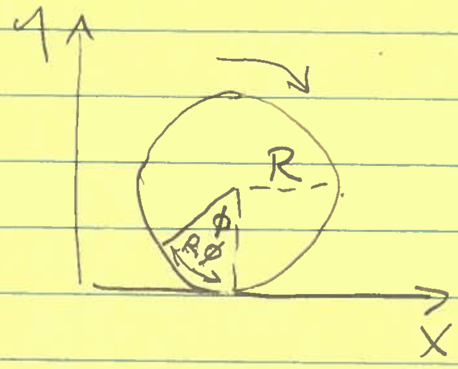
1) q_i are Natural if $\vec{r}_\alpha = \vec{r}_\alpha(q_1, .. q_N)$ (Cartesian) is ind of time

2) # degrees of freedom $\equiv$ # q_i that can be independently varied — small displacements from start

3) If # deg. freed. = # q_i system is holonomic

Idea of 3 : holonomic vs. nonholonomic

Ex. a) disk rolling w/o slipping on x-axis



$x = R\phi$ in. cond.

$v - R\dot{\phi} = 0$ $x(\phi=0)=0$

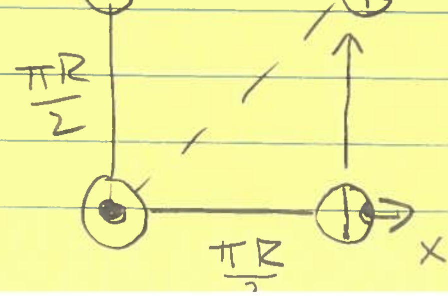
state described by x or ϕ

2D space but $\left\{ \begin{array}{l} 1 \text{ degree of freedom} \\ 1 \text{ generalized coordinate} \end{array} \right.$

Constraint "without slipping" ties x ; ϕ together

Ex b. sphere rolling in xy plane

mark on N pole



3b) cont'd: So if we try to describe state of system by x, y alone we can't do it: no unique relationship between x, y and θ, φ $\Rightarrow$ can't eliminate angles

$\Rightarrow$ Nonholonomic constraint # deg. free. = 2

gen. coordinates = 5 = 2 + 3 Euler angles

Restrict ourselves to holonomic constraints

Ex 1 pendulum

$$\mathcal{L} = T - U$$
$$= \frac{1}{2} m\ell^2 \dot{\phi}^2 - mg\ell(1-\cos\phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mg\ell\sin\phi$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt} m\ell^2\dot{\phi} = m\ell^2\ddot{\phi}$$

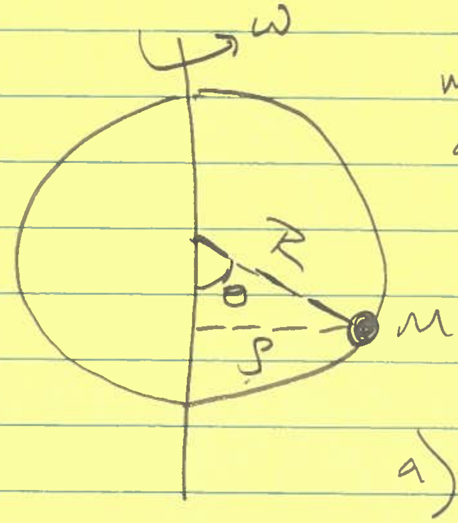
$$\Rightarrow \quad \ddot{\phi} + \frac{g}{\ell}\sin\phi = 0$$

$$\simeq \ddot{\phi} + \frac{g}{\ell}\phi = 0$$

$\Rightarrow$ small oscillations frequency $\omega^2 = g/\ell$

Ex 2 N.B. Lagrange's eqns. work in noninertial frames provided $\mathcal{L}$ formulated in inertial frame.

Ex. 2 cont'd Bead on rotating wire circle



$\rho = R\sin\theta$

m constrained to move on a circle rotated w/ angular velocity ω.

Intuition?

a) ω very slow ⇒ bead sits at $\theta = 0$. Small oscillations around this point

b) ω very fast ⇒ bead sits at $\theta = \pi/2$, small oscillations take place there

Q: what happens for ω in between?

$$T = \frac{1}{2} m R^2 \left(\underbrace{\dot\theta^2}_{\text{tangential}} + \underbrace{\omega^2 \sin^2\theta}_{\text{normal}} \right)$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} m R^2 \left(\dot\theta^2 + \omega^2 \sin^2\theta \right) - mgR(1-\cos\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} = mR^2\ddot\theta$$

$$\Rightarrow \quad \ddot\theta = \left(\omega^2\cos\theta - \frac{g}{R} \right)\sin\theta$$

looks like pendulum eqn. with "θ-dependent osc. freq."

Q: what is eq. position of bead for given ω? "Equilibrium" $\Rightarrow$ $\ddot{\theta} = 0$ so

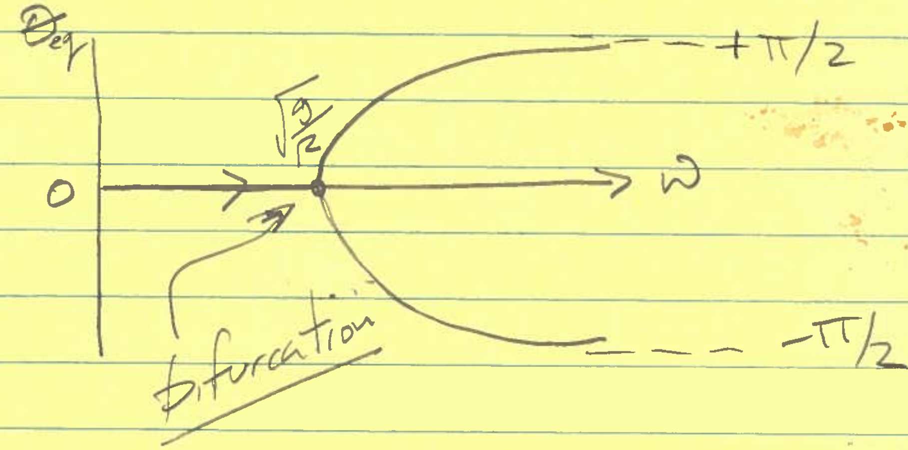
$\boxed{\sin\theta = 0}$ or $\boxed{\cos\theta = \frac{g}{R\omega^2}}$

θ or π soln only if $\frac{g}{R\omega^2} \leq 1$

So for a) ω very fast $\theta_{eq} = \cos^{-1}\frac{g}{R\omega^2}$

b) ω very slow $\theta = 0$

Note $\theta = 0$ only soln until $\omega^2 = \frac{g}{R}$!



Remarks: a) Note to completely specify position of bead, need to specify azimuthal angle φ. But $\mathcal{L} \neq \mathcal{L}(\varphi)$, so no add'l $\mathcal{L}$-eqn. (ignorable or 'cyclic' coordinate)

b) ρ is not an independent generalized coordinate since bead's position on hoop specified already by θ.