8/29/22

Announcements

HW 1 due Sept. 7 (pwd ok?)
Ofc. hrs PH T,W 4PM Chao Zhang M 5pm

Last time: review of Lagrange's eqns

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \; ; \quad \mathcal{L} = T - U \text{ for conservative holonomic systems}$$

Examples: a) simple pendulum, b) bead on rotating circular wire.

N.B. For b) we found eqn. of motion for bead:

$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta$$

$\ddot{\theta} = 0$ Eq. solns: $\sin\theta = 0$ or $\cos\theta = \frac{g}{R\omega^2}$ if $\omega^2 > \frac{g}{R}$

Q: why isn't $\sin\theta = 0$ a soln. for $\omega^2 > \frac{g}{R}$
A: it is, but we have for small θ

$$\ddot{\theta} = \left(\omega^2\left(1 - \frac{\theta^2}{2} \cdots\right) - \frac{g}{R}\right)\left(\theta - \frac{\theta^3}{3} + \cdots\right)$$

$$\cong \underbrace{\left(\omega^2 - g/R\right)}_{\text{if } \omega^2 > g/R \implies \text{no restoring force!}}\theta$$

Q: what about $\theta = \pi$ → always unstable
hint: $\sin\theta \cong -(\theta - \pi) + \cdots$

LAGRANGIAN REVIEW CONT'D
8/27/21

Symmetries and Conservation Laws

Emmy Noether (1882-1935)
b. Erlangen, Germany, studied math
Invited by D. Hilbert and F. Klein to
join faculty at U Göttingen,
Rest of faculty refused → Privatdozent.
1933 → Bryn Mawr, Princeton IAS.

* Noether's theorem (1918) For every continuous
symmetry of a mechanical system ∃
a conservation law.
* major contribution to abstract algebra

Examples: transl. sym $\Rightarrow$ momentum cons.
time transl sym $\Rightarrow$ energy cons.

Method of proof: transform Lagrangian
under some symm. operation, show
that $\delta \mathcal{L} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0$

→ conserved

Momentum: shift coordinates by const $\vec{\varepsilon}$
$x_i \to x_i + \varepsilon_i \qquad \dot{x}_i \to \dot{x}_i \Rightarrow T \to T$

Transl. inv. $\Rightarrow U(\vec{r} + \vec{\varepsilon}) = U(\vec{r})$

example — grav. forces among N particles

$$U = -\sum_{i \neq j} \frac{G m_i m_j}{|\vec{r}_i - \vec{r}_j|} \xrightarrow[\vec{r}_i \to \vec{r}_i + \vec{\varepsilon}]{} \text{same}$$

So $\delta T = 0$, $\delta U = 0 \Rightarrow \delta \mathcal{L} = 0$

Let x-coord. of all N particles change by ε :

$$\delta \mathcal{L} = \varepsilon \sum_i \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

$$= \varepsilon \sum_i \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \varepsilon \frac{d}{dt} \sum_i \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}_i}}_{P_{x_i}}$$

$\Rightarrow$ total momentum $\vec{P} = (P_x, P_y, P_z)$ conserved.

Other remarks on Lagrangian method

1) Dissipative forces must be included explicitly: $\frac{\partial \mathcal{L}}{\partial x} + F_{fric} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}$ (7.12)

2) Method of Lagrange multipliers to include constraints
   a) express $\mathcal{L}$ in terms of all coordinates
   b) constraint equation $f(x_1, x_2, \ldots) = 0$

Then solve modified eqns

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}, \quad \text{etc.}$$

N+1 eqns with N+1 unknowns $\longrightarrow$ solve

3) Lagrangian for charge q in $\vec{E}, \vec{B}$ fields

$$\boxed{\begin{aligned}\vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t}\\ \vec{B} &= \vec{\nabla} \times \vec{A}\end{aligned}}$$

$$\mathcal{L}(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2} m \dot{\vec{r}}^{\,2} - q\left(V - \dot{\vec{r}} \cdot \vec{A}\right)$$

$$\Rightarrow \quad \ddot{\vec{r}} = q\left(\vec{E} + \dot{\vec{r}} \times \vec{B}\right)$$

Chapter 13. Hamiltonian Mechanics

Consider Lagrangians that do <u>not</u> depend explicitly on time, and consider total t-derivative

$$\frac{d}{dt}\mathcal{L}(q_1 \ldots q_N ; \dot{q}_1 \ldots \dot{q}_N) = \underbrace{\sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i}_{1} + \underbrace{\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i}_{2} + \cancel{\frac{\partial \mathcal{L}}{\partial t}}$$

1 use $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \dot{p}_i$

where $p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ "canonical momentum conjugate to q_i" "generalized momentum" (same things)

2 $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i = p_i \ddot{q}_i$

$$\Rightarrow \frac{d}{dt}\mathcal{L} = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i)$$

$$= \frac{d}{dt}\sum_i (p_i \dot{q}_i)$$

$$\Rightarrow \frac{d}{dt}\underbrace{(p_i \dot{q}_i - \mathcal{L})}_{H \text{ "Hamiltonian"} = T+V} = 0$$

Restatement:

"time translational invariance" of $\mathcal{L}$
($t \to t + \varepsilon$ ε = const leaves $\mathcal{L}$ invariant)

$$\Rightarrow \boxed{\frac{d}{dt} H = 0}$$

i.e. Total energy conserved

(2nd example of Noether's theorem!)

$\mathcal{L} \to H$ is an example of Legendre transformation

$$\mathcal{L} = \mathcal{L}(q, \dot{q}) \qquad H = H(q, p)$$

p is momentum "conjugate to" q

Math: function of 2 vars. $f(x,y)$

$$df = \underbrace{\left(\frac{\partial f}{\partial x}\right)_y}_{u} dx + \underbrace{\left(\frac{\partial f}{\partial y}\right)_x}_{v} dy$$

$$= u\,dx + v\,dy \qquad (*)$$

u, x conjugate to each other
v, y " "

Now notice $d(yv) = y\,dv + v\,dy$
Subtract from (*) to get

$$d(\underbrace{f - yv}_{g}) = u\,dx - y\,dv$$

g is Legendre transform of f
we switched from $f(x,y)$ to $g(x,v)$
where v is conjugate of y.

Aside Familiar example from thermodynamics:

Fixed quantity of gas, independent variables S, V

$U = U(S,V)$ Internal energy $dU = T\,dS - P\,dV$
T, P conjugate to S, V respectively

Legendre transform to quantity w/ independent variables S, P

$H = H(S,P)$ Enthalpy $dH = T\,dS + V\,dP$

where $\left.\frac{\partial U}{\partial S}\right|_V = T \qquad \left.\frac{\partial U}{\partial V}\right|_S = -P$