8/31/22

Announcements

HW 1 Sept. 7
Ofc. Hrs 4pm T, W (PH)
5pm R (Zhang)

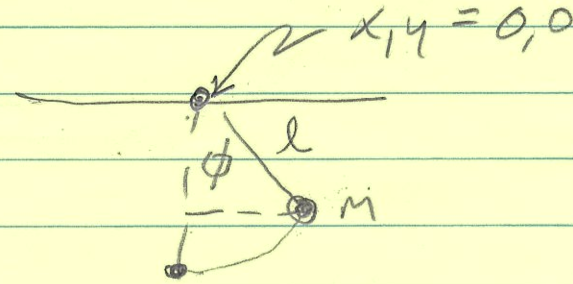
<u>Last time</u>

Noether thm : continuous sym. => cons. law
showed for p-cons.

Dissipation in Lagrangian systems

Method of Lagrange multipliers

<u>Ex:</u> <u>Simple pendulum</u>



① easy way

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$\Rightarrow \quad \ddot{\phi} = \frac{-g}{l}\sin\phi$$

② hard way: coords x, y constraint

$$x, y = l\sin\phi, -l\cos\phi$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$$

Construct $\frac{\partial \mathcal{L}}{\partial x} = \ldots$, $\frac{\partial \mathcal{L}}{\partial y} = \ldots$ <u>But</u> what about constraint?

$\textcircled{0'}$

Lagrange multipliers $\mathcal{L} \to \mathcal{L} + \lambda f(x,y)$

constraint: $f(x,y) \equiv x^2 + y^2 - \ell^2$

3 eqns: $\frac{\partial \mathcal{L}}{\partial x} = \ldots \frac{\partial \mathcal{L}}{\partial y} = \ldots \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow f = 0$

should get same answer $\ddot{\phi} = -g/\ell \sin\phi$

N.B. ① remember now $\frac{\partial \mathcal{L}}{\partial x} = \lambda 2x, \ldots$
$\frac{\partial \mathcal{L}}{\partial y} = \lambda 2y, \ldots$

Soln. (not covered in class)

$\frac{\partial \mathcal{L}}{\partial x} = \ldots \quad \Rightarrow \quad m\ddot{x} = \lambda 2x$

$\frac{\partial \mathcal{L}}{\partial y} = \ldots \quad \Rightarrow \quad m\ddot{y} = -mg + \lambda 2y$

constraint $\quad x^2 + y^2 = \ell^2$

eliminate λ: $\quad \lambda = \frac{m\ddot{x}}{2x}$

solve:

$\ddot{y} = (\ell \sin\phi \dot{\phi})^{\cdot} = \ell\cos\phi\dot{\phi}^2 + \ell\sin\phi\ddot{\phi}$

$\ddot{x} = (\ell\cos\phi\dot{\phi})^{\cdot} = -\ell\sin\phi\dot{\phi}^2 + \ell\cos\phi\ddot{\phi}$

$(\cancel{\ell\cos\phi\dot{\phi}^2} + \ell\sin\phi\ddot{\phi}) = -g + (\cancel{-\ell\sin\phi\dot{\phi}^2} + \ell\cos\phi\ddot{\phi})\left(\frac{-\ell\cos\phi}{\ell\sin\phi}\right)$

simplifies to $\ell\ddot{\phi} = -g\sin\phi$

Nomenclature for Hamiltonian systems

$\mathcal{L}$ { configuration space $\{q_1, \ldots q_N\}$

state space $\{q_1 \ldots q_N ; \dot{q}_1 \ldots \dot{q}_N\}$

specifying pt. in state spaces at time t
$\Rightarrow$ unique trajectory determined by
Lagrange's eqns.

generalized (conjugate, canonical) momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

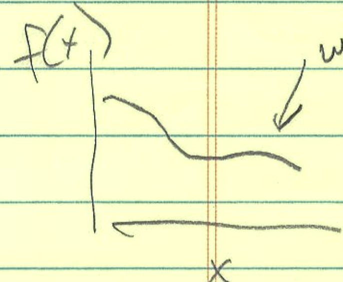
need not have dimensions of physical p!

$\mathcal{H}$ { phase space $\{q_1 \ldots q_N ; p_1 \ldots p_N\}$

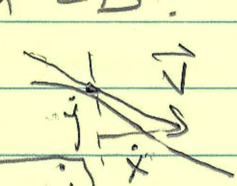
1D systems

assume $\mathcal{L} = T(q, \dot{q}) - U(q)$

Reminder Q: how can T depend on q as well as $\dot{q}$?
A: recall bead moving on wire in 2D!



$$T = \frac{1}{2} m \left(1 + f'(x)^2\right) \dot{x}^2$$



$$\boxed{f'(x) = \frac{\dot{y}}{\dot{x}}}$$

why? $\vec{v} = (\dot{x}, \dot{y})$

So "in general" write $T = A(q)\dot{q}^2$

why only $\propto \dot{q}^2$ — pf. ?
makes intuitive sense at least

$$H = p\dot{q} - \mathcal{L}$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = 2A(q)\dot{q} \Rightarrow \dot{q} = \frac{p}{2A(q)} \equiv \dot{q}(p,q)$$

$$\boxed{\Rightarrow H = p\dot{q}(q,p) - \mathcal{L}}$$

$\mathcal{L}$ should be considered as function of p, q

(1) Consider $$\frac{\partial H}{\partial q} = \cancel{p\frac{\partial \dot{q}}{\partial q}} - \left[\frac{\partial \mathcal{L}}{\partial q} + \cancel{\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial q}}\right]$$

L's eqn: $$= -\frac{\partial \mathcal{L}}{\partial q} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = -\dot{p}$$

(2) Consider $$\frac{\partial H}{\partial p} = \dot{q} + \cancel{p\frac{\partial \dot{q}}{\partial p}} - \cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p}}$$

$\frac{\partial q}{\partial p} = 0$
q, p ind

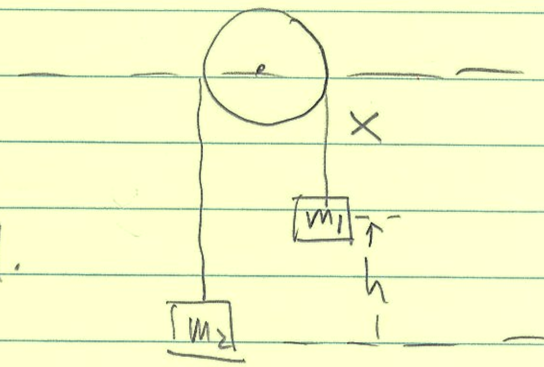
$$= \dot{q}$$

$\therefore$ Hamilton's eqns

$$\boxed{\begin{aligned}\frac{\partial H}{\partial q} &= -\dot{p} \\ \frac{\partial H}{\partial p} &= \dot{q}\end{aligned}}$$

equivalent to L's eqns?

Ex. 1 Atwood machine



$$x + y = l$$

Use x as generalized coord.

$$T = \frac{1}{2}(m_1+m_2)\dot{x}^2$$

$$U = -(m_1 - m_2) g x \qquad h = \text{const} - x$$

Assume $H = T + U$ (true when q_i are natural)
(exception: see 13.11)

"natural": relation between $\vec{r}_\alpha$ and $\vec{q}$ does not involve time.

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m_1+m_2)\dot{x}$$

Need $H(p,q) \Rightarrow \dot{x} = \frac{p}{m_1+m_2}$

$$T = \frac{p^2}{2(m_1+m_2)}$$

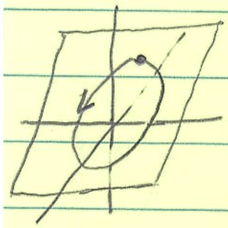
$$\Rightarrow \boxed{H = \frac{p^2}{2(m_1+m_2)} - (m_1-m_2) g x}$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m_1+m_2} \qquad \dot{p} = -\frac{\partial H}{\partial x} = (m_1-m_2) g$$

$$\ddot{x} = \frac{\dot{p}}{m_1+m_2} = \frac{(m_1-m_2)g}{m_1+m_2} \quad \text{std. ans.}$$

Ex. 2 Central force

★ Assertion: if particle moves under influence of central force $\vec{F}(\vec{r})$, its motion is confined to a plane! (Ch. 8 — will prove later)



So we can choose our plane to be parametrized by (r, ϕ)

$$T = \frac{1}{2} m \big(\underbrace{\dot{r}^2}_{\text{radial}} + \underbrace{r^2 \dot{\phi}^2}_{\text{transverse}} \big) \qquad U = U(r)$$

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \qquad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$\Rightarrow \boxed{H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + U(r)}$$

Ham: $\displaystyle \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2}$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial}{\partial r} U$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \text{cons. of a.m.}$$

Note on radial eqn: $\dot{r} = p_r/m \Rightarrow m\ddot{r} = \dot{p}_r$

$$m\ddot{r} = -\frac{\partial U}{\partial r} + p_\phi^2/(mr^3) \quad \text{"centrifugal force"}$$