9/12/22

Announcements

HW 2 due Sept. 19 Lectures recorded, link on web.

Quiz 2 on Wednesday Sept. 14

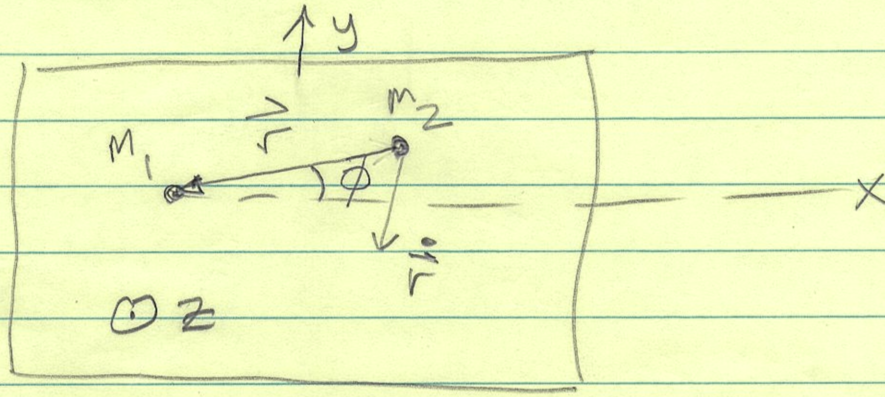
Discussion of quiz, test dates

Last time central force problems

$\mathcal{L} \neq \mathcal{L}(\phi) \Rightarrow L$ conserved $\Rightarrow$ motion in a plane

$l = \mu r^2 \dot{\phi}$

$l = L_z = \mu \vec{r} \times \dot{\vec{r}}$



$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$$

$\boxed{\phi}$

A.M. cons. : $\dfrac{\partial \mathcal{L}}{\partial \phi} = 0 = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dfrac{d}{dt}\underbrace{\left(\mu r^2 \dot{\phi}\right)}_{l = L_z}$

(ϕ ignorable)

Check that $\mu r^2 \dot{\phi}$ corresponds to L_z :

$$\vec{r} = r(\cos\phi, \sin\phi)$$

$$\dot{\vec{r}} = \dot{r}(\cos\phi, \sin\phi) + r(-\sin\phi, \cos\phi)\dot{\phi}$$

$$\vec{r} \times \dot{\vec{r}} = r^2\dot{\phi}\underbrace{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}}_{1} = r^2\dot{\phi} \quad \checkmark$$

So ℓ is constant of motion, $\dot\phi = \frac{\ell}{\mu r^2}$

$\boxed{r}$ $$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot r} \Rightarrow$$

$$\mu \ddot r = -\frac{dU}{dr} + \mu r \dot\phi^2$$

$$\boxed{\mu \ddot r = -\frac{dU}{dr} + \frac{\ell^2}{\mu r^3}}$$

1D, 1-body eqn. from 3D, 2-body eqn, when ℓ is specified!

NB: $\frac{\ell^2}{\mu r^3} > 0$ has dimensions of a force F_{cf}

Not a new physical force, but shows up in effective radial eqn. — "centrifugal force"
Sometimes convenient to write as gradient of its own fictitious potential $F_{cf} = -\frac{dU_{cf}}{dr}$

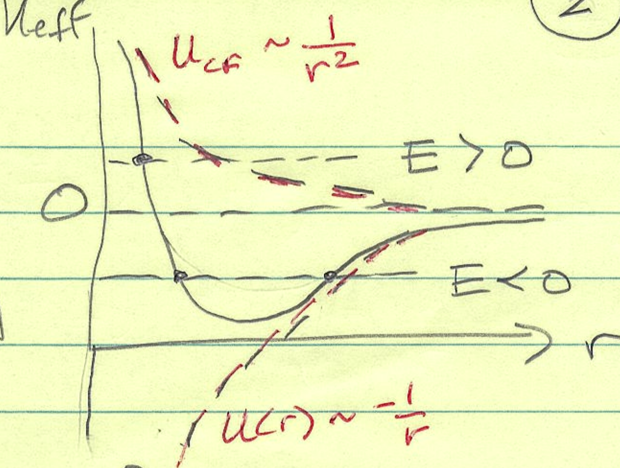
$$U_{cf} = \frac{\ell^2}{2\mu r^2}$$

$$U_{eff} = U(r) + \frac{\ell^2}{2\mu r^2}$$

$$\Rightarrow \quad \mu \ddot r = -\frac{dU_{eff}}{dr}$$

Plot $U_{eff}(r)$
for $U(r) = \frac{-G m_1 m_2}{r}$

NB zero of potential at $r = \infty$!



Energy $T + U = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r) = \text{const.} \equiv E$

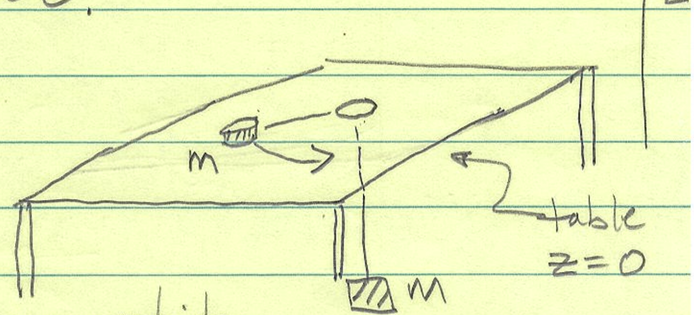
$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{\ell^2}{2\mu r^2}}$$

$E>0$ or $E<0$ determines class of soln. to differential eqn.

$E > 0$ one crossing pt. $\Rightarrow$ unbound orbit
$E < 0$ two " " bound orbit

NB Book does only Kepler problem with $U \sim \frac{1}{r}$, quite special in some ways. But above discussion is general for central force.

Ex. 1



Physics 1 : $T = Mg$

$T = \frac{mv^2}{r} \Rightarrow$ circular orbit

w/ $v^2/r = g \Rightarrow \omega = \dot{\theta} = \sqrt{g/r}$

Gravity at earth's surface:

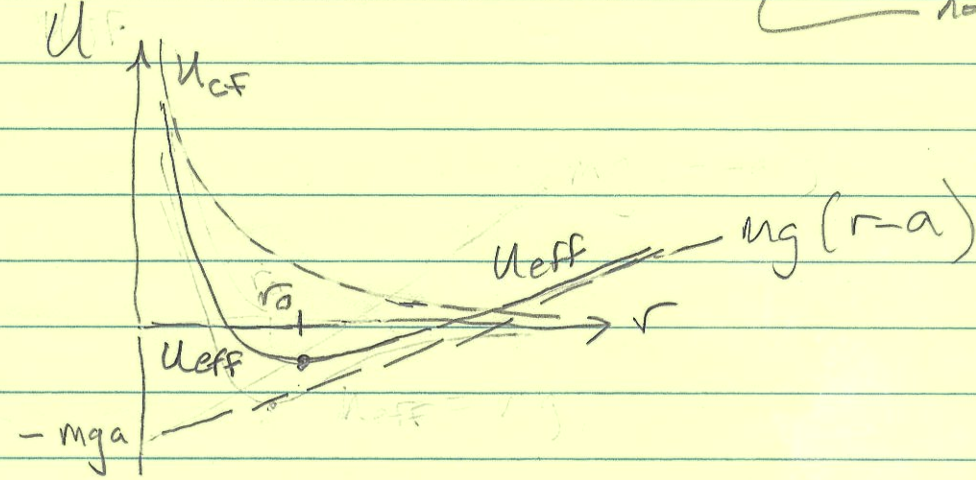
$$U = mgz = mg(r-a)$$

where a is length of string $(r-a<0)$

Force is central, a.m. conserved

$\mu = \frac{m}{2}$ $\Rightarrow$ $U_{eff} = mg(r-a) + \frac{\ell^2}{2mr^2}$

not μ!



We get our Physics I circular soln by solving for r_0 (r where U_{eff} is minimum)

$\ell = mr_0^2\omega$

$$\frac{\partial U_{eff}}{\partial r} = 0 = mg - \frac{\ell^2}{mr_0^3} = m\left(g - \frac{\omega^2}{r_0}\right)$$

$$\Rightarrow \omega = \sqrt{g/r_0} \text{ again}$$

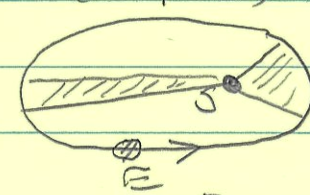
But now we see there are other solutions, other types of orbits with $E > E_0$!

These will not be conic sections since $U(r) \not\propto \frac{1}{r}$

"Kepler problem" ("$1/r^2$ problem")

Johannes Kepler 1571 – 1630

- Sought God in perfection of mathematics
- Obsessed w/ problem of planetary motion
- Math teacher Graz, Austria –
  Planetary orbits' radii as radii
  of nested platonic solids
- Looked for better data – joined
  Tycho Brahe in Prague, Oranienborg
- Discovered 3 laws of planetary motion
  a. orbits are conic sections (ellipses)
  b. equal areas in equal times
  c. $T^2 \sim R^3$



- 1st to ask why planets moved
  as they did. His theory: magnetism!!
- his work essential to Newton's discovery
  of gravity $1/r^2$ law

$$U(r) = -\frac{G m_1 m_2}{r}$$

so $$U_{eff} = -\frac{G m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}$$