9/14/22

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Announcements

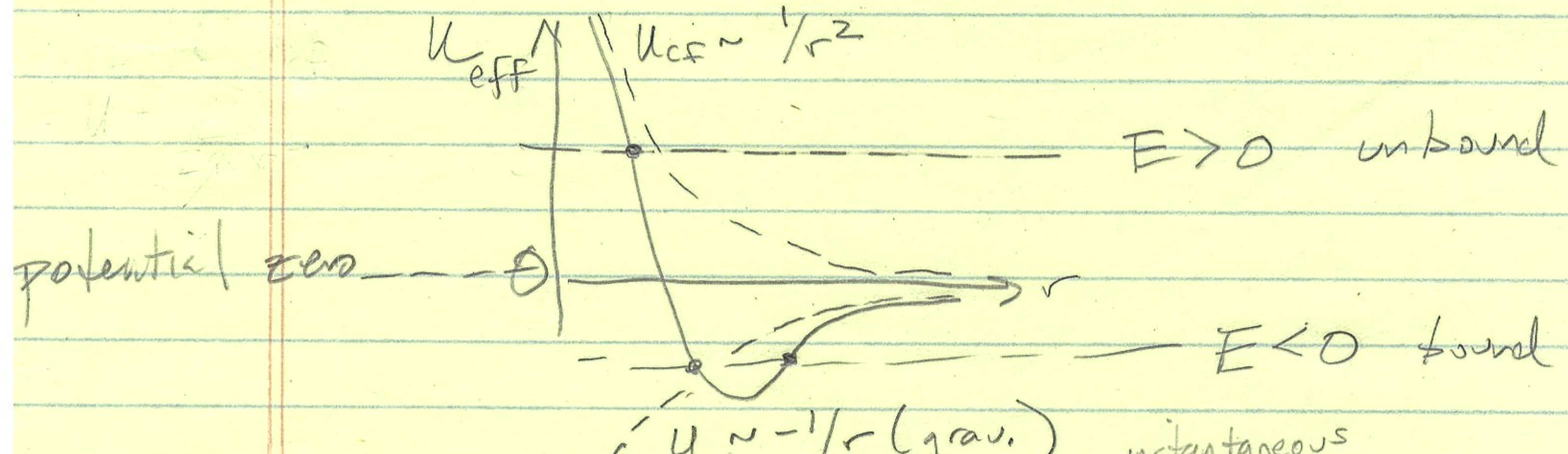
- HW2 due 9/19
- Test date? Proposal 9/30

<u>Last time</u> Central force problem
Effective 1D equation for fixed $l = \mu r^2 \dot{\phi}$

Radial eqn. of motion

$$\boxed{\mu \ddot{r} = -\frac{dU}{dr} + \frac{l^2}{\mu r^3}}$$

$$U_{eff} = U + U_{cf} = U + \frac{l^2}{2\mu r^2}$$

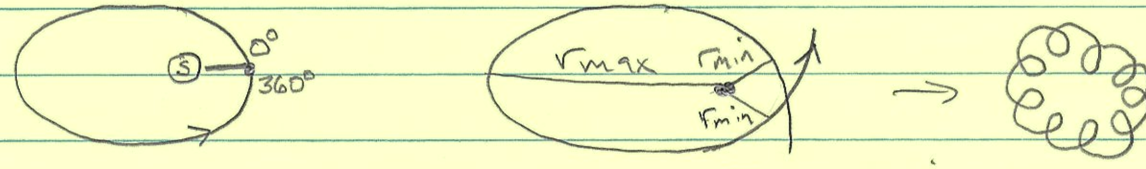


When $U = E$ $T_r = 0$? no instantaneous <u>radial</u> motion

Equation of orbit: want eqn for $r(\phi)$

Remark on solution:

Planet moves around sun in ellipse w/sun at one focus. Has unique property that when ϕ goes through 2π, planet returns to same point, $r_{min} \to r_{max} \to r_{min}$ in 2π



$r_{min} \to r_{max} \to r_{min}$ in $< 2\pi$

$U \sim 1/r$ $\qquad$ $U \sim 1/r^\alpha \quad \alpha \neq 1$

Start w/ radial eqn. of motion

want to find $r(\phi)$?

$$\mu \ddot{r} = -\frac{dU}{dr} + \frac{l^2}{\mu r^3}$$

$$= \underbrace{-\frac{G m_1 m_2}{r^2}}_{F} + \frac{l^2}{\mu r^3}$$

convenient new variable $\boxed{u = \frac{1}{r}}$

$\boxed{l = \mu r^2 \dot{\phi}}$

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{l}{\mu r^2}\frac{d}{d\phi} = \frac{l u^2}{\mu}\frac{d}{d\phi}$$

note $\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{l u^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right) = \frac{-l}{\mu}\frac{du}{d\phi}$

$$\Rightarrow \ddot{r} = \frac{l u^2}{\mu}\frac{d}{d\phi}\left(\frac{-l}{\mu}\frac{du}{d\phi}\right) = \frac{-l^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2}$$

$$-\frac{l^2 u^2}{\mu}\frac{d^2 u}{d\phi^2} = -Gm_1m_2u^2 + \frac{l^2u^3}{\mu}$$

$$\boxed{u'' = -u + \frac{Gm_1m_2\mu}{l^2}} \qquad u = u(\phi)$$

NB important that last term = const.
works only for $\frac{1}{r^2}$ Force!

put $\quad u'' = -u + \frac{\gamma\mu}{l^2} \qquad \gamma = Gm_1m_2$

$$w \equiv u - \frac{\gamma\mu}{l^2} \implies w'' = -w$$

soln $\quad w = A\cos(\phi - \delta)$

choice of initial $\delta = 0$ when $\phi = 0$

$$\implies u = \frac{1}{r} = \frac{\gamma\mu}{l^2} + A\cos\phi$$

$$\equiv \underbrace{\frac{\gamma\mu}{l^2}}_{\equiv c^{-1}}(1 + \varepsilon\cos\phi)$$

$\boxed{c = \frac{l^2}{\gamma\mu}}$

dimensionless $\boxed{\varepsilon \equiv \frac{Al^2}{\gamma\mu}}$ called eccentricity > 0

$$\implies \boxed{r = \frac{c}{1 + \varepsilon\cos\phi}}$$

NB $r(\phi)$ periodic w/ period 2π
($1/r^2$ force)

Bounded orbits $E < 0$, $\varepsilon < 1$

$\varepsilon = 1$ must be some kind of critical value, since if $\varepsilon < 1$ r never diverges, hence orbit is bounded

$$-1 \le \cos\phi \le 1 \implies \underbrace{\frac{c}{1+\varepsilon}}_{r_{min}} \le r \le \underbrace{\frac{c}{1-\varepsilon}}_{r_{max}}$$

$[r = r_{min},\ \phi = 0]$ perihelion

$[r = r_{max},\ \phi = \pi]$ aphelion

Q: Is this soln. an ellipse?

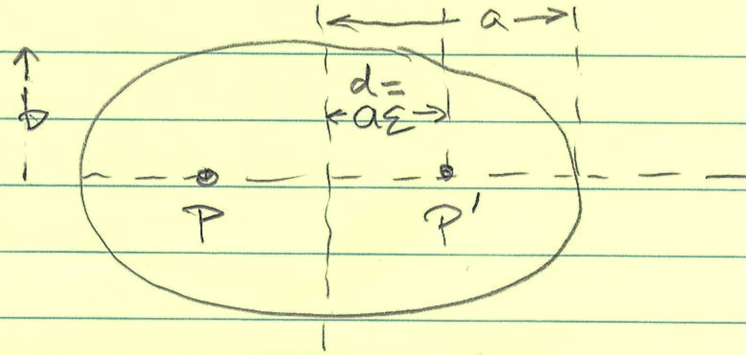
Cartesian coords $\vec{r} = r(\phi)(\cos\phi, \sin\phi)$

$$x^2 = r^2\cos^2\phi = \frac{c^2\cos^2\phi}{(1+\varepsilon\cos\phi)^2}$$

$$y^2 = r^2\sin^2\phi = \frac{c^2\sin^2\phi}{(1+\varepsilon\cos\phi)^2}$$

Note: $x^2 + y^2 = r^2 \ne$ const. unless $\varepsilon = 0$ (circle)

Ellipse geometry



Show (tedious - problem 8.16):

$$a = \frac{c}{1-\varepsilon^2}; \quad b = \frac{c}{\sqrt{1-\varepsilon^2}}; \quad d = a\varepsilon$$

Orbit eqn. is $\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$

is ellipse w/ x-coord shifted by $a\epsilon$

<u>Remark on constants of motion</u>

* parameters of orbit fixed by ℓ or E.

$$\boxed{c = \frac{\ell^2}{\gamma\mu}}$$

Can we find $E(\ell)$?

since $E = \frac{1}{2}\mu\dot{r}^2 - \frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$

not obvious how to relate. But since it's a const, can calculate at a turning point, e.g. r_{min}, where $\dot{r}=0$ so $E = U_{eff}$.

r_{min} from ellipse geometry is

$$r_{min} = a(1-\varepsilon) = \frac{c}{1+\varepsilon} = \frac{\ell^2}{\mu\gamma(1+\varepsilon)}$$

$\Rightarrow$ find $E(\ell)$