9/16/22

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Announcements: Hw 2 due Sept 19

Test 1 → Sept 30. chs 8, 12, 13

Term paper prospectus due Oct. 5

Last time

Solution for "shape" of orbit in Kepler problem $r(\phi)$

Derived diff eq. for $u = 1/r$

$$u'' = -u + \frac{\gamma\mu}{\ell^2}$$

$c = \ell^2/\gamma\mu$

$\varepsilon = A\ell^2/\gamma\mu$ = "eccentricity"

$$u = \frac{\gamma\mu}{\ell^2}(1 + \varepsilon\cos\phi)$$

$$\boxed{r = \frac{c}{1+\varepsilon\cos\phi}}$$

Bounded orbits $E<0$, $\varepsilon < 1$

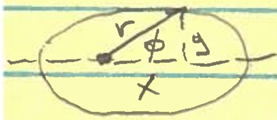
$\varepsilon = 1$ must be critical pt. since if $\varepsilon < 1$ r never diverges ⇒ orbit bounded

$$-1 \le \cos\phi \le 1 \Rightarrow \underbrace{\frac{c}{1+\varepsilon}}_{r_{min}} \le r \le \underbrace{\frac{c}{1-\varepsilon}}_{r_{max}}$$

$r = r_{min}$, $\phi = 0$ "perihelion"

$r = r_{max}$, $\phi = \pi$ "aphelion"

Q: is solution an ellipse?



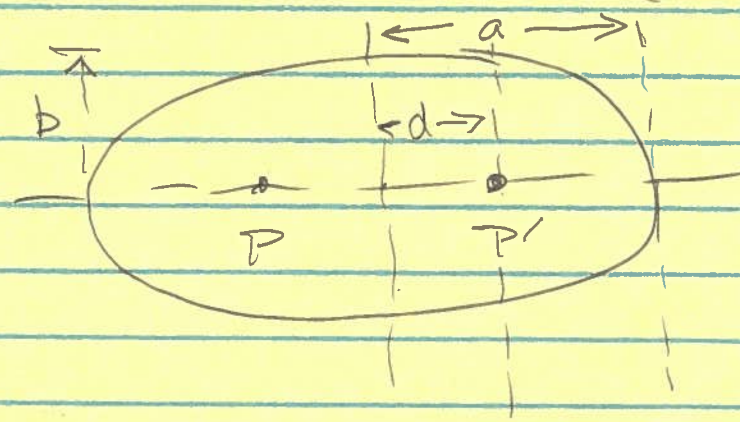
Cartesian coords $\vec{r} = r(\phi)\,[\cos\phi, \sin\phi]$

$$x^2 = r^2\cos^2\phi = \frac{c^2\cos^2\phi}{(1+\varepsilon\cos\phi)^2}$$

$$y^2 = r^2\sin^2\phi = \frac{c^2\sin^2\phi}{(1+\varepsilon\cos\phi)^2}$$

NB: $x^2+y^2 = r^2 \neq$ const unless $\varepsilon = 0$ (circle

<u>Ellipse geometry</u>



Check:

$$\boxed{a = \frac{c}{1-\varepsilon^2} \quad b = \frac{c}{\sqrt{1-\varepsilon^2}} \quad d = a\varepsilon}$$

<u>Orbit</u> $\dfrac{(x+d)^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Ellipse!

<u>Energy</u> $E = \frac{1}{2}\mu\dot{r}^2 - \frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$

Turning points r_{min}, r_{max} $\dot{r} = 0$

$$\Rightarrow E = \frac{-\gamma}{r_{min}} + \frac{\gamma^2}{2\mu r_{min}^2}$$

$r_{min} = \dfrac{c}{1+\varepsilon}$ substitute

$$\Rightarrow \quad E = \frac{\gamma^2 \mu}{l^2}\left(-(1+\varepsilon) + \frac{(1+\varepsilon)^2}{2}\right)$$

$$= \frac{\gamma^2 \mu}{2l^2}\left(\varepsilon^2 - 1\right) < 0$$

$\Rightarrow$ E for bound orbit < 0 depends only on l
bigger $l \Rightarrow$ smaller E (larger negatively!)

$\Rightarrow$ you can fix l <u>or</u> E

but border between $l_{unbound}$, l_{bound} not clear

<u>Unbound orbits</u> $E \geq 0$

(A) From expression for E, we see

$$E \rightarrow \frac{\gamma^2 \mu}{l^2}\left(\varepsilon^2 - 1\right) \quad \text{at } r_{min}$$

$= 0$ when $\varepsilon = 1$ <u>marginal case</u>

However gen'l solution for $E < 0$ doesn't work:
e.g. $a = \frac{c}{1-\varepsilon^2} \rightarrow \infty$.

Go back to $r = \dfrac{c}{1 + \varepsilon \cos\phi}$

If $\varepsilon = 1$, $r(\phi = \pi) \rightarrow \infty$ unbound

Solution for orbit is (tedious)

$$y^2 = c^2 - 2cx \quad \text{parabola}$$

B $E > 0$, 1 turning point, $\varepsilon > 1$

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1 \quad \text{hyperbola}$$

$$\alpha = \frac{c}{\varepsilon^2 - 1} \quad \beta = \frac{c}{\sqrt{\varepsilon^2 - 1}} \quad \delta = \varepsilon\alpha$$

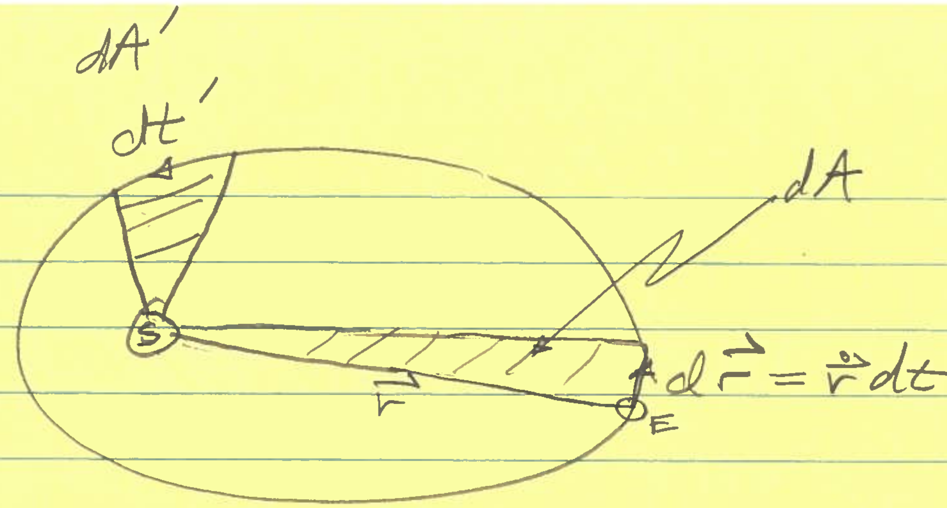
<u>Summary</u>: orbits are <u>conic sections</u>!

$\varepsilon = 0$	$E < 0$	circle
$0 < \varepsilon < 1$	$E < 0$	ellipse
$\varepsilon = 1$	$E = 0$	parabola
$\varepsilon > 1$	$E > 0$	hyperbola

☆ Voila: Newton's soln to Kepler's problem. K's 1st law is special case

Q 2nd + 3rd laws?

2nd law



$$\frac{dA}{dt} = \frac{1}{2}\frac{\vec{r}\times\dot{\vec{r}}\,dt}{dt} = \frac{\ell}{2\mu} = \text{const!}$$

3rd law Since $\dot{A}$ = const, we may say

$$\tau = \text{period} = \frac{A}{\dot{A}} = \frac{2\mu}{\ell}(\pi ab)$$

Recall

$a = \dfrac{c}{1-\varepsilon^2}$

$b = \dfrac{c}{\sqrt{1-\varepsilon^2}}$

$c = \dfrac{\ell^2}{\gamma\mu}$

$$= \frac{2\mu c^2\pi}{\ell(1-\varepsilon^2)^{3/2}}$$

$$\Rightarrow \tau^2 = \frac{4\pi^2\mu^2}{\ell^2}c\left(\frac{c}{(1-\varepsilon^2)}\right)^3 = \underbrace{\left(\frac{4\pi^2\mu}{\gamma}\right)}_{\text{const,}}a^3$$

$$\tau^2 \propto a^3 \;!$$

not just for $\varepsilon=0$!

Note "const." is not quite constant

$$\frac{\mu}{\gamma} = \frac{m_1m_2/m_1+m_2}{Gm_1m_2} \sim \frac{1}{m_1+m_2} = \frac{1}{M_S+m_{planet}}$$

solar system

$\Rightarrow \tau^2/a^3$ is a little different for each planet!