9/19/22

Announcements

HW 3 posted tomorrow, due 9/24
1 extra credit computational problem
Read Ch. 12 next

Last time: summary of Kepler problem:

proved	$E<0$	$\varepsilon=0$	circle
Kepler's 3 laws	$E<0$	$\varepsilon<1$	ellipse
from $1/r^2$ force	$E=0$	$\varepsilon=1$	parabola
	$E>0$	$\varepsilon>1$	hyperbola

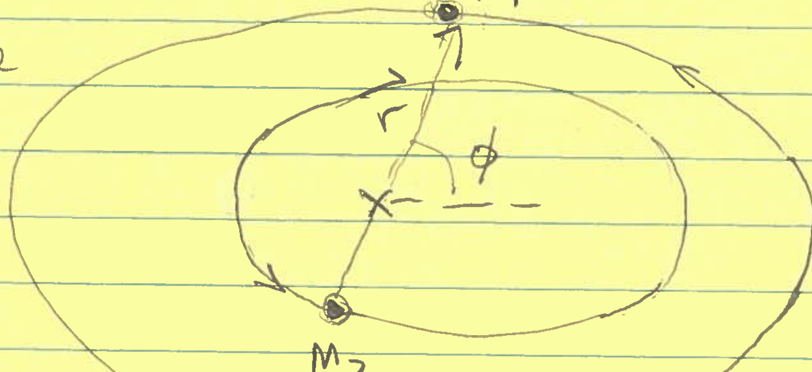
Orbits are conic sections <u>only</u> for $\frac{1}{r^2}$ force!

4. Planetary orbits are ellipses (or circles) even when viewed from CM frame.

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

set $\vec{R}=0$

2 bodies comparable mass



$$\vec{r} = \frac{c}{1+\varepsilon\cos\phi}$$

Examples

1. What happens to Earth orbit (assume circular) if 1/2 of sun's mass suddenly vaporized by new Klingon superweapon? Is Earth still bound to sun?

$$\text{Earth's}\quad PE \quad U = U_0/2$$
$$KE \quad T = T_0$$

$$E = T + U = T_0 + \tfrac{1}{2}U_0$$

Easy way:

$$\boxed{\begin{array}{l} \text{Recall \underline{virial Theorem}:} \\ \text{general} \quad \overline{T} = -\tfrac{1}{2}\overline{\sum_i \vec{F}_i \cdot \vec{r}_i} \\ 1/r^2 \text{ force} \quad \overline{T} = -\tfrac{1}{2}\overline{U} \end{array}}$$

$\overline{A} \equiv$ time avg. of A

$\Rightarrow$ for circular orbits $T = -\tfrac{1}{2}U$

before: $T_0 + U_0 = \tfrac{1}{2}U_0 < 0$

after: $T_0 + \tfrac{1}{2}U_0 = -\tfrac{1}{2}U_0 + \tfrac{1}{2}U_0 = 0$

$\Rightarrow$ orbit is parabolic (just unbound)

2 Satellite has min height $h_{min} = 300\,km$ (perigee), max $h_{max} = 3000\,km$ (apogee). Find orbit's eccentricity.

$$r_{min} = R_E + h_{min} \qquad r_{max} = R_E + h_{max}$$

$$RE = 6300\,km$$

$$r_{min} = \frac{c}{1+\varepsilon} \qquad r_{max} = \frac{c}{1-\varepsilon} \qquad c = ?$$

$$r_{max} - r_{min} = \frac{c\,2\varepsilon}{1-\varepsilon^2} \qquad r_{max} + r_{min} = \frac{2c}{1-\varepsilon^2}$$

$$\boxed{\frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \varepsilon} = \frac{9300 - 6600}{9300 + 6600} = 0.17$$

see p. 6

3 What do orbits of individual planets look like in CM frame?

$r(\phi)$ is an ellipse, say $E < 0$
but what are $r_1(\phi_1)$, $r_2(\phi_2)$?

Easy: $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$

Take $\vec{R} = 0$ at rest

w/o loss of generality take $\vec{R} = 0$

$\vec{r}_{1,2}(\phi) = \pm\frac{m_{2,1}}{M}\vec{r}(\phi)$ ellipses w CM at focus

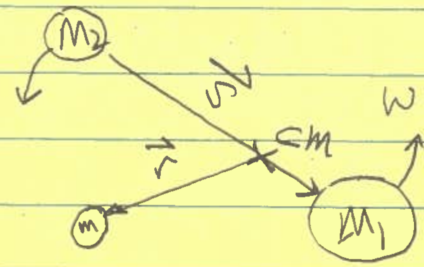
N.B. 2-body problem exactly soluble for arb. $U(r)$, may be complicated.

But $\boxed{\text{3-body problem}}$ generally insoluble

One subproblem of great interest can be solved — "restricted 3-body prob."

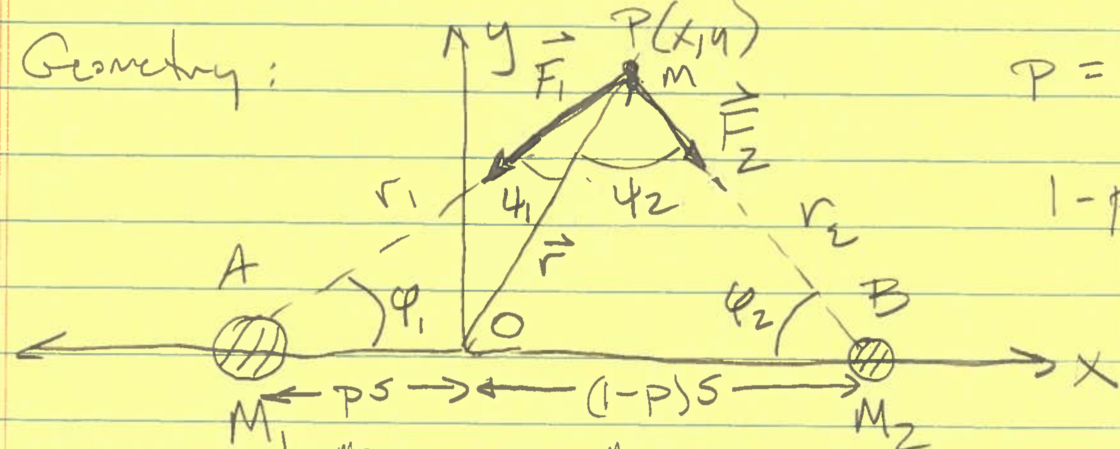
- Start with 2 large masses M_1, M_2 orbiting each other: e.g. S-E, or E-M systems
- Add small mass m assumed not to perturb orbits of big masses.

Q: are there equilibrium points for 3rd body in gravitational field of other 2?



Assume M_1, M_2 executing circular orbits around CM.

Geometry:



$\frac{M_2}{M}s$ $\qquad$ $\frac{M_1}{M}s$

$$p = \frac{M_2}{M}$$

$$1-p = \frac{M_1}{M}$$

For m to be in equilibrium, it must be rotating CM of M_1 + M_2 at exact same angular speed,

$$\boxed{m\vec{r}\omega^2 + \vec{F}_1 + \vec{F}_2 = 0}$$ no net eff. force

$$F_1 = \frac{GmM_1}{r_1^2} \qquad F_2 = \frac{GmM_2}{r_2^2}$$

Decompose net force along $\hat{r}$ direction, $\perp$ forces must cancel; $\parallel$ forces $= m\omega^2 r$

$\perp$ $\quad F_1^\perp = F_1 \sin\psi_1 = F_2^\perp = F_2 \sin\psi_2$

$$\Rightarrow \frac{F_1}{F_2} = \frac{\sin\psi_2}{\sin\psi_1} \qquad (*)$$

Law of sines applied to ΔOAP

$$\frac{\sin\varphi_1}{r} = \frac{\sin\psi_1}{ps}; \quad \sin\varphi_1 = \frac{y}{r_1}$$

$$\Rightarrow \sin\psi_1 = \frac{ps\sin\varphi_1}{r} = \frac{psy}{rr_1}$$

Similarly for ΔOBP

$$\frac{\sin\varphi_2}{r} = \frac{\sin\psi_2}{(1-p)s}; \quad \sin\varphi_2 = \frac{y}{r_2}$$

$$\Rightarrow \quad \sin\psi_2 = \frac{(1-p)sy}{rr_2}$$

Use $(*)$:

$$\frac{F_1}{F_2} = \frac{\frac{GmM_1}{r_1^2}}{\frac{GmM_2}{r_2^2}} = \frac{\sin\psi_2}{\sin\psi_1} = \frac{\frac{(1-p)sy}{rr_2}}{\frac{psy}{rr_1}}$$

$$\frac{M_1 r_2^2}{M_2 r_1^2} = \frac{r_1(1-p)}{r_2 p}$$

$$\left(\frac{r_2}{r_1}\right)^3 = \frac{M_2(1-p)}{M_1 p} = \frac{\frac{M_1M_2}{M}}{\frac{M_2M_1}{M}} = 1$$

so $\boxed{r_1 = r_2 \equiv \rho \text{ is one solution } (A)}$

* But transverse forces $F_1^\perp$, $F_2^\perp$ also cancel if $\sin\psi_1 \stackrel{!}{=} \sin\psi_2 = 0$

e.g. $\psi_1 \to 0$, $\psi_2 \to \pi \Rightarrow \boxed{y = 0 \; (B)}$

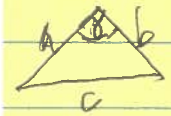
Now balance radial forces

$$mr\omega^2 = F_1\cos\psi_1 + F_2\cos\psi_2 \quad (**)$$

$r_1 = r_2 = \rho$

$$\boxed{\cos\varphi_1 = \frac{r^2+\rho^2 - p^2 s^2}{2r\rho}} \quad \text{a}$$

law of cosines



$c^2 = a^2 + b^2 - 2ab\cos\gamma$

$$\boxed{\cos\varphi_2 = \frac{r^2+\rho^2 - (1-p)^2 s^2}{2r\rho}} \quad \text{b}$$

Note M_1, M_2 going instantaneously in circles, so we must have

force on M_1: $\dfrac{GM_1M_2}{s^2} = M_1(ps)\omega^2$

$p = \dfrac{M_2}{M}$

$$\Rightarrow \boxed{\omega^2 = \frac{GM_2}{ps^3} = \frac{GM}{s^3}} \quad \text{c}$$

Substitute a, b, c into **:

simplifies to $\rho^3 = \left(\dfrac{r^2+\rho^2 - ps^2 + p^2 s^2}{2r^2}\right)^3 s$ **

More geometry (remember $r_1 = r_2 = \rho$)

$\cos\varphi_1 = \dfrac{s}{2\rho}$ OAP $r^2 = \rho^2 + p^2 s^2 - 2\rho p s\cos\varphi_1$

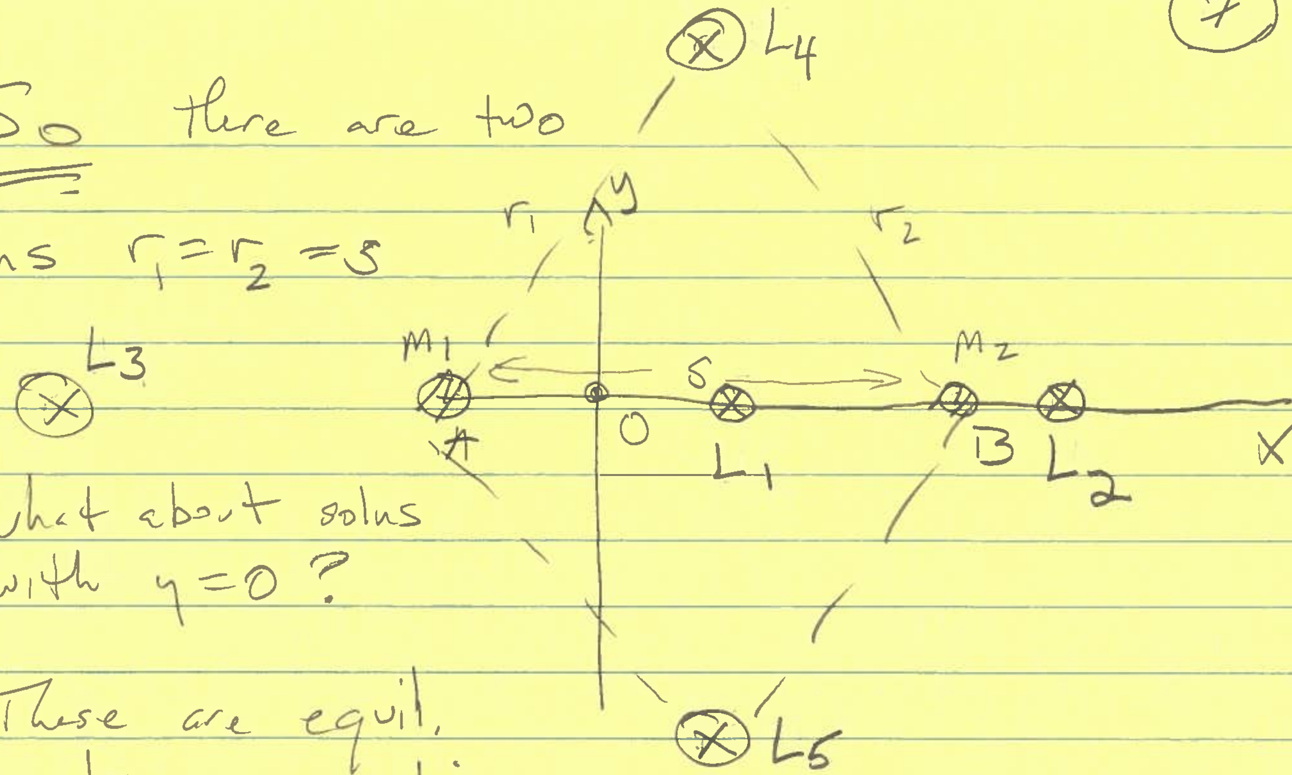
$\Rightarrow r^2 = \rho^2 + p^2 s^2 - ps^2$

substitute into ** $\Rightarrow \boxed{\rho = s}$

So APB is equilateral triangle!

(A) So there are two

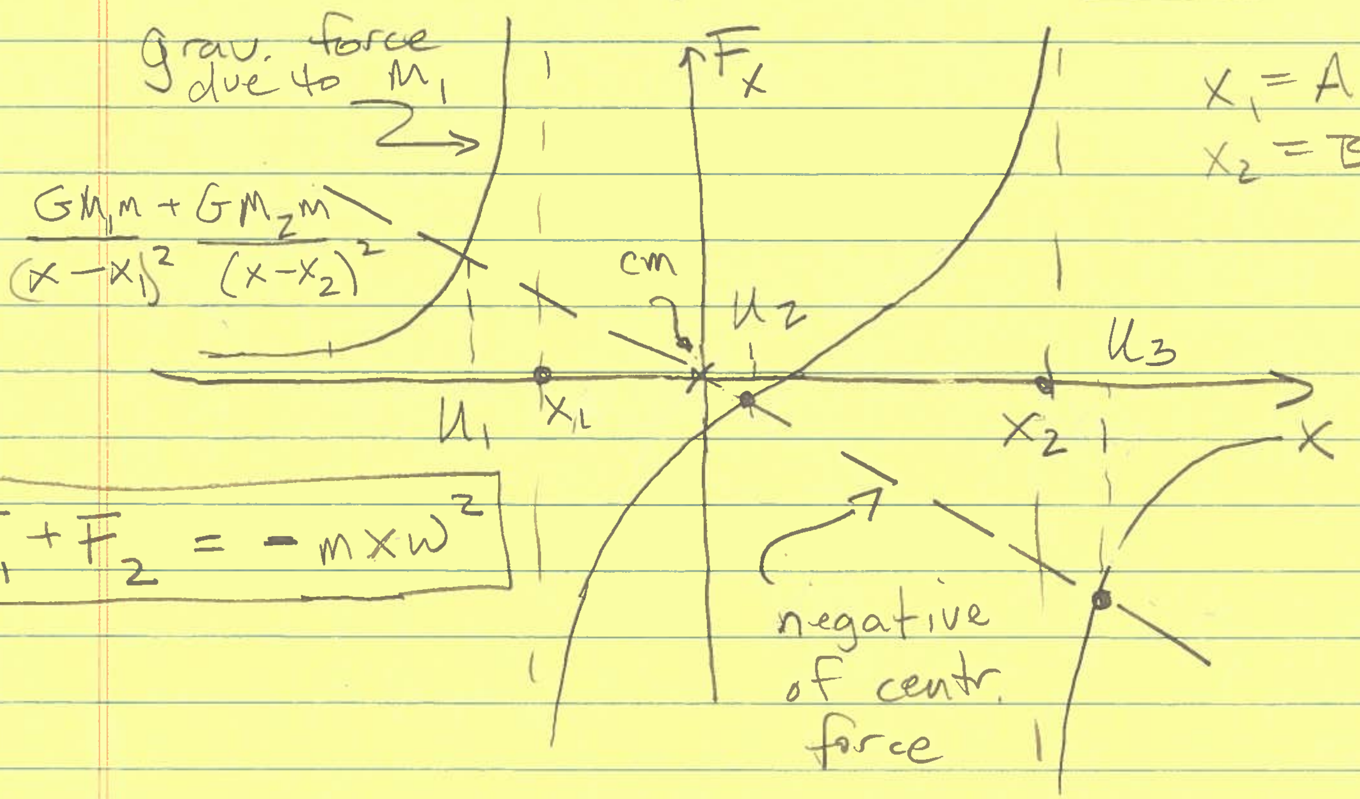
solns $r_1 = r_2 = s$



(B) what about solns with $y=0$?

These are equil. points on a line but in the presence of centrifugal force!

Plot gravitational force F_x on a small test mass along x axis $+ \Rightarrow$ to right



$$\boxed{F_1 + F_2 = -mx\omega^2}$$

Equilibrium points L_1, L_2, L_3, L_4, L_5 are called "Lagrange pts."

L_4, L_5 are stable*
L_1, L_2, L_3 are unstable (saddle pts.)

(To show this, need to calculate

$$\overset{=}{\nabla} U_{eff}(\vec{r}) \quad \text{complicated})$$

actually they are <u>maxima</u> of U_{eff}, but coriolis force forces sideways velocity, stable orbits possible

(for another day!)

- James Webb telescope orbits at L_2 point. Why?

- <u>Trojan asteroids</u> at stable Lagrange pt. of Jupiter-Sun system

- LISA (Laser Interferometer Space Antenna) Pathfinder launched to L1 point "Technology demonstration mission" for main LISA mission (2034)