9/21/22

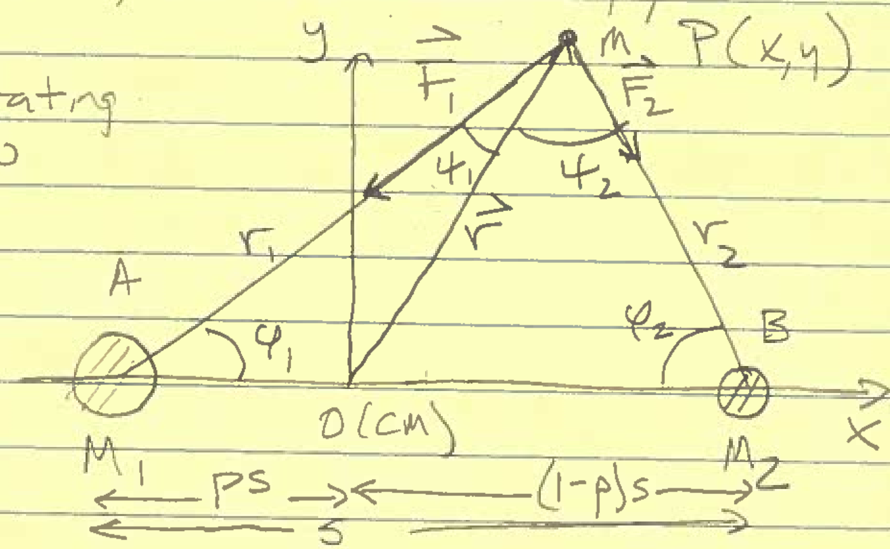
Announcements

HW3 posted due 9/28 HW2 solns posted
Reading: Ch. 12.1 - 12.4
Quiz Friday 9/23
Term paper abstract due 10/5

Last time: examples, restricted 3-body problem

Geometry in rotating frame, ang. vel ω

$p = M_2/M$
$1-p = M_1/M$
$M = M_1 + M_2$



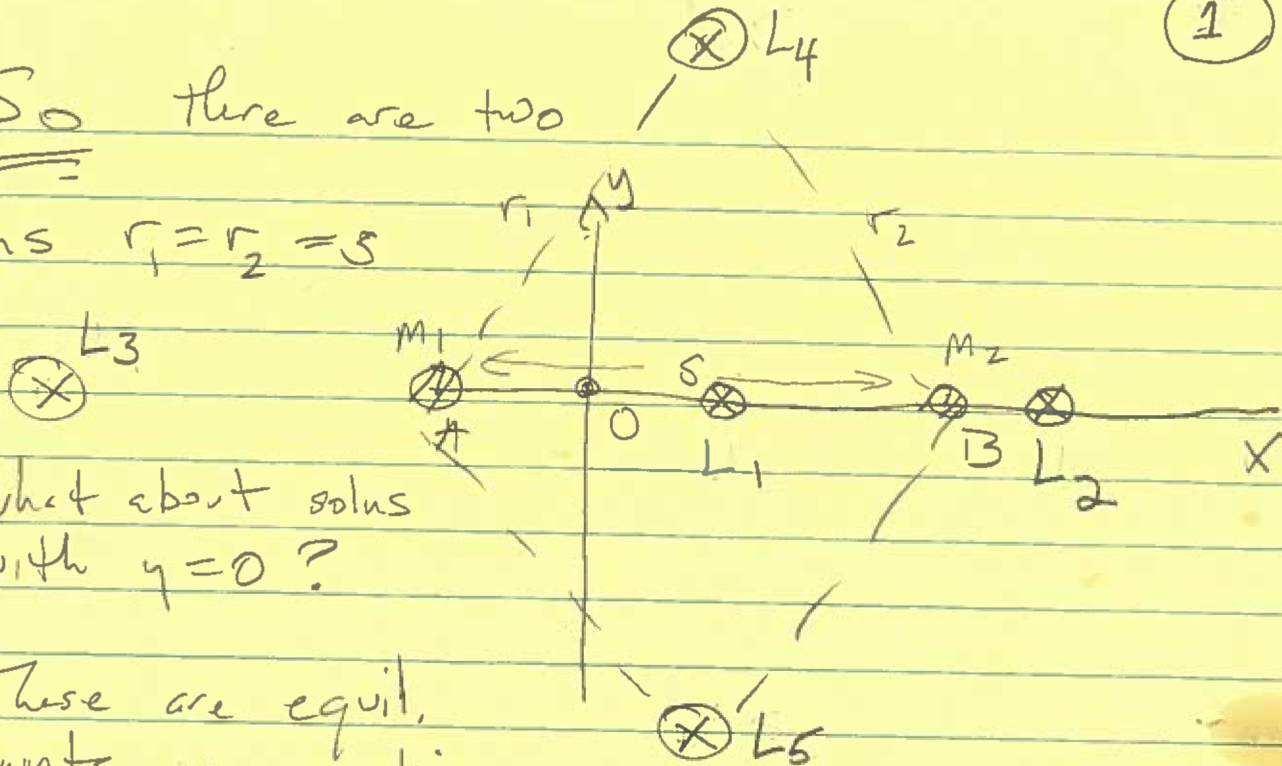
Proved that either $r_1 = r_2 \equiv \rho$ (A)

or if $\psi_1 = 0$, $\psi_2 = \pi$ $y = 0$ (B)

More geometry for (A)-type solutions
$\Rightarrow \boxed{\rho = s} \Rightarrow$ equilateral triangle!

see notes 9/19/22

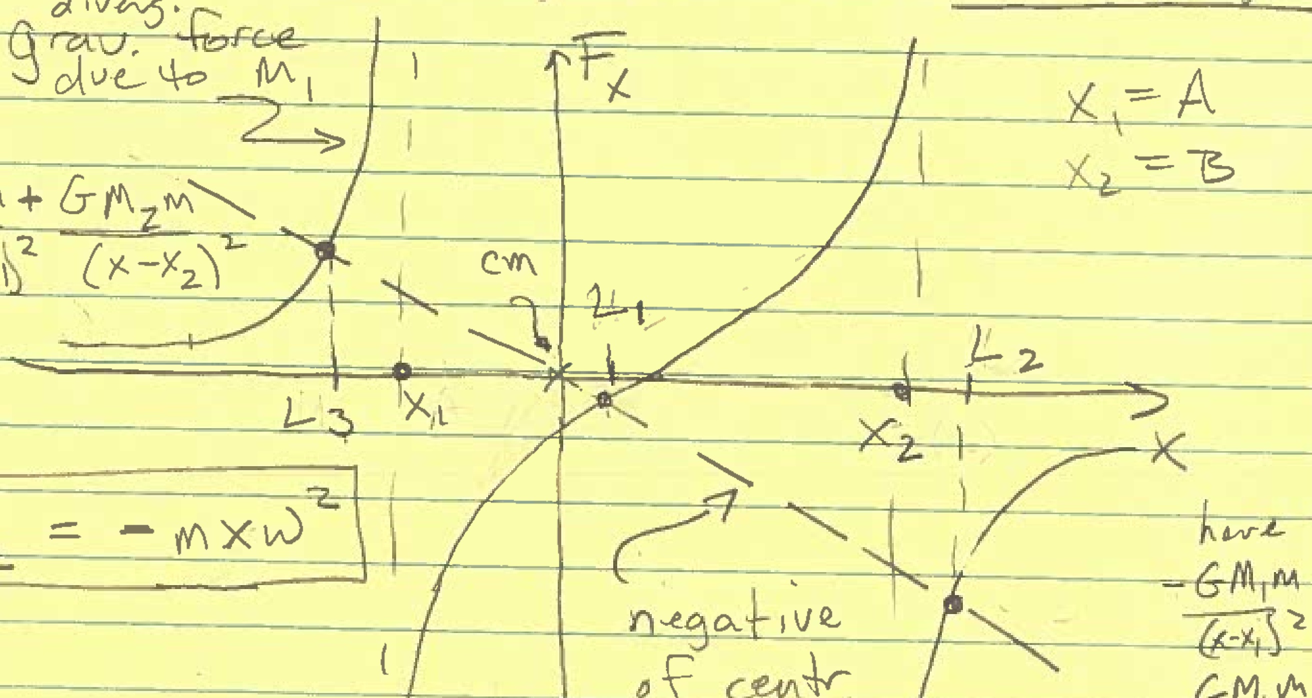
A So there are two solns $r_1 = r_2 = s$



B what about solns with $y=0$?

These are equil. points on a line but in the presence of centrifugal force!

Plot gravitational force F_x on a small test mass along x axis $+ \Rightarrow$ to right



$$\frac{GM_1m}{(x-x_1)^2} + \frac{GM_2m}{(x-x_2)^2}$$

$$-\frac{GM_1m}{(x-x_1)^2} - \frac{GM_1m}{(x-x_2)^2}$$

$$\boxed{F_1 + F_2 = -mx\omega^2}$$

This region $-\frac{GM_1m}{(x-x_1)^2} + \frac{GM_2m}{(x-x_2)^2}$

Equilibrium points L_1, L_2, L_3, L_4, L_5
are called "Lagrange pts."

L_4, L_5 are stable*
L_1, L_2, L_3 are unstable (saddle pts.)

(To show this, need to calculate

$\overset{=}{\nabla} U_{eff}(\vec{r})$ complicated)

actually they are maxima of U_{eff},
but coriolis force forces sideways
velocity, stable orbits possible

(for another day!)

- James Webb telescope orbits at L_2 point.
  Why?

- Trojan asteroids at stable Lagrange
  pt. of Jupiter - Sun system

- LISA (Laser Interferometer Space Antenna)
  Pathfinder launched to L1 point
  "Technology demonstration mission"
  for main LISA mission (2034)

9/21/22 Nonlinear mechanics + chaos (3)

Announcements

Read 12.1–4, 12.9

HW3 due Friday Sept. 24 — secs. 12.1-4, 12.9.

Test 1 Monday, Sept. 27 Chs 8, 12, 13 *

Test 1 review Friday Sept. 24

Test + HW pages to be passwd protected "mechanics" "phy4222#!"

History of Chaos Field

- late 19th century Henri Poincaré studies 3-body-problem, shows in some circumstances soln v. sensitive to initial conditions
- little interest til 60's–70's of 20th cent. computer solns to nonlinear eqns. became available
- we cover one tiny aspect: "period doubling" route to chaos in one nonlinear system

Math reminder: linear, nonlinear, homogeneous, inhomogeneous ODE's

function $x(t)$, linear eqn has no powers of x^α, $\dot{x}^\alpha$... other than $\alpha = 1$

"nth order" ODE $\Rightarrow$ derivatives up to $\frac{d^n}{dt^n}$

<u>homogeneous</u> ODE has <u>no</u> term with no $x, \dot{x}, \ddot{x}$ ...

We are interested in <u>nonlinear</u>, <u>inhomogeneous</u> ODEs because they exhibit chaos

<u>Examples</u>

lin, hom. ODEs $\quad m\ddot{x} = -kx$

$a\dddot{x} = ax + b\dot{x}$

lin, inhom ODEs $\quad m\ddot{x} = -kx + A\cos\omega t$

$m\ddot{x} = -kx + Bt$

nonlin, hom ODE $\quad m\ddot{x} = -k\sin(ax)$

e.g. pendulum $\quad mL^2\ddot{\phi} = -mgL\sin\phi$

(reduces to linear eqn for <u>small</u> osc.)

$\sin\phi \simeq \phi + \ldots \quad mL^2\ddot{\phi} = -mgL\phi$

nonlin, inhom ODE. e.g.

<u>DDP</u>: driven, damped pendulum

$$\boxed{mL^2\ddot{\phi} = \underbrace{-mgL\sin\phi}_{\text{nonlinearity}} \underbrace{-bL^2\dot{\phi}}_{\substack{\text{damping}\\ \text{(linear)}}} + \underbrace{AF(t)}_{\substack{\text{inhomogeneity:}\\ \text{("driving force")}}}}$$

No superposition principle for NL ODEs

reminder: $m\ddot{\phi} = -k\phi$

- If ϕ is soln, $a\phi$ is too, w/ a=const
- If ϕ_1 is soln, ϕ_2 soln, then

$$a\phi_1 + b\phi_2 \text{ also soln}$$

$$\text{e.g. } (\sin\omega t, \cos\omega t), (e^{i\omega t}, e^{-i\omega t})$$

- also true for inhom eqns. (sort of)

take 2 solns ϕ_1, ϕ_2

$$m\ddot{\phi}_1 = -k\phi_1 + A\sin\omega t$$
$$m\ddot{\phi}_2 = -k\phi_2 + A\sin\omega t$$

$$m(\ddot{\phi}_1 + \ddot{\phi}_2) = -k(\phi_1 + \phi_2) + 2A\sin\omega t$$

$$\Rightarrow m\left(\frac{\ddot{\phi}_1 + \ddot{\phi}_2}{2}\right) = -k\left(\frac{\phi_1 + \phi_2}{2}\right) + A\sin\omega t$$

$\Rightarrow \frac{1}{2}(\phi_1 + \phi_2)$ is soln particular comb.

Not true for nonlinear eqns.

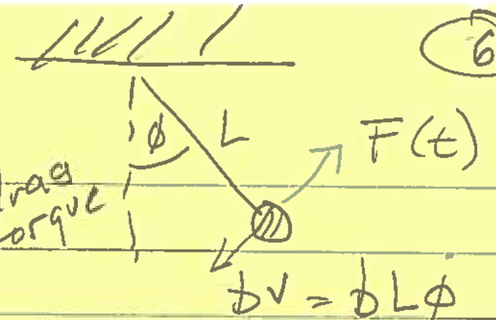
e.g. $$m\ddot{\phi} = -c\phi^2$$

ϕ_1, ϕ_2

$$m\ddot{\phi}_1 = -c\phi_1^2$$
$$m\ddot{\phi}_2 = -c\phi_2^2$$

$$m(\ddot{\phi}_1 + \ddot{\phi}_2) = -c(\phi_1^2 + \phi_2^2) \neq -c(\phi_1 + \phi_2)^2$$

Back to our DDP



standard form

$$mL^2\ddot{\phi} = -bL^2\dot{\phi} - mgL\sin\phi + LF(t)$$

drag torque

torque from drive

torque from grav

torque $= I\alpha = mL^2\ddot{\phi}$

Take $F(t) = F_0 \cos\omega t$

Divide by mL^2

$$\boxed{\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\,\omega_0^2 \cos\omega t}$$

$$2\beta = \frac{b}{m}\,; \quad \omega_0^2 = \frac{g}{L}\,; \quad \gamma = \frac{F_0}{mg}$$

Dimensions $[\beta] = 1/\text{time}$, $[\omega_0] = 1/\text{time}$

$$[\gamma] = 1$$

Case 1 Driven oscillation in linear regime

$$\sin\phi \simeq \phi \quad \text{near} \quad \phi = 0$$

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\phi = \gamma\,\omega_0^2\cos\omega t$$

after transient (depends on β), oscillation is at driving freq, ω (may be out of phase)

$$\phi(t) \underset{t\to\infty}{=} A\cos(\omega t - \delta)$$

Case 2   small nonlinearity

$$\sin\phi \cong \phi - \frac{1}{3!}\phi^3 \qquad \phi \text{ near } 0$$

$$\Rightarrow \ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\left(\phi - \frac{1}{6}\phi^3\right) = \gamma\omega_0^2\cos\omega t$$

solution $\phi(t)$ should be close to linear one

$\hat{=}$ adding new term $\propto \cos^3(\omega t-\delta)$

$$= \frac{1}{4}\left(\cos 3(\omega t-\delta) + 3\cos(\omega t-\delta)\right)$$

$\Rightarrow$ new small term $\propto \cos 3(\omega t-\delta)$

$$\Rightarrow \phi(t) \cong A\cos(\omega t-\delta) + \underbrace{B\cos^3(\omega t-\delta)}_{\text{higher harmonic}}$$

In general, nonlinear terms generate higher harmonics at integer multiples of drive frequency $n\omega$.