9/7/22

Announcements

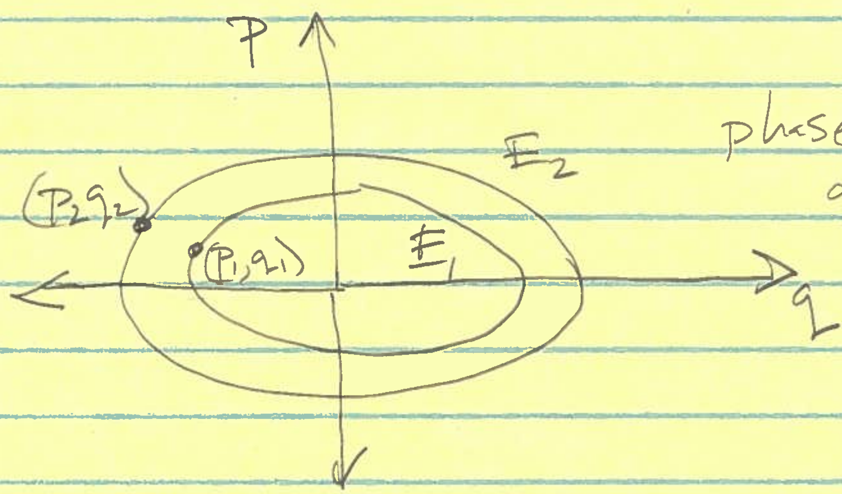
HW1 due today by 4pm, HW2 due Sept. 19
PH ofc. hr. today 3pm 2156

Q1 today

Last time

- Motion of particle confined to a cone — circular, elliptical orbits

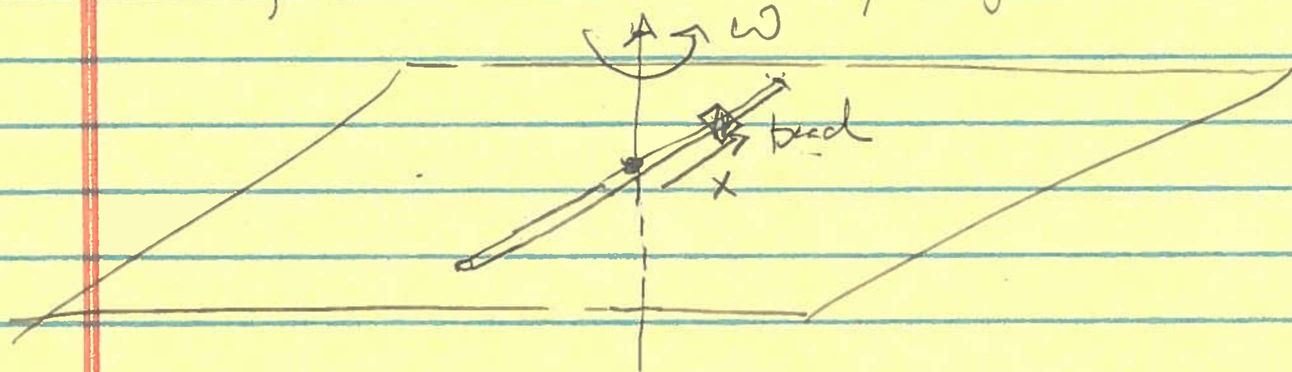
- phase space trajectories for Hamiltonian systems, e.g. $H = \frac{P^2}{2m} + \frac{1}{2}kx^2$ SHO



phase space orbits do not cross for same H, different initial conds.

Non-natural coords. $H \neq T+U$

<u>Example</u>: bead on frictionless, straight rod, forced to spin w/ ang. vel. ω:



(1) Inertial (lab) frame:

$$T = \frac{1}{2}m\left(\dot{x}^2 + x^2\omega^2\right) \qquad \mathcal{L} = T$$

gen. mom. $p = \dfrac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$

$$H = p\dot{x} - \mathcal{L} = \frac{p^2}{2m} - \frac{1}{2}m x^2 \omega^2$$

but this is <u>not</u> the energy

a) lab frame $E = T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m x^2 \omega^2$

b) rot. frame $E = \frac{1}{2}m\dot{x}^2$

non-inertial frames always non-natural.

Exercise: solve H's eqns to find $x \propto x_0 e^{\omega t}$

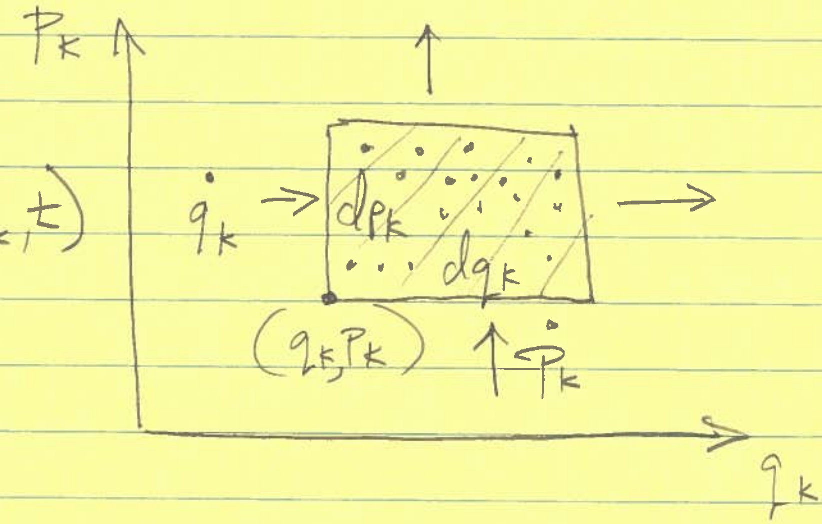
Liouville's Theorem

A set of points near each other in phase space $\{\vec{p}, \vec{q}\}$ may be thought of as a) one physical system but with slightly different initial conditions, or b) many particles moving acc. to same H. Imagine pts. very close, such that we can define a <u>density</u> of phase sp. pts. ρ

$$\# \text{ pts within volume } V \quad dN = \rho\, dV$$

$$dV = dq_1 \ldots dq_n \, dp_1 \ldots dp_n$$

$$\rho = \rho(p_k, q_k, t) = \frac{dN}{dp_k\, dq_k}$$



As time evolves, what happens to the density inside the box? Note this can change because more points flow in or out, or if dq_k, dp_k themselves increase in size.

\# pts. moving through left-hand edge in dt:

$$\rho \frac{dq_k}{dt} dp_k = \rho \dot{q}_k dp_k, \quad \text{and similarly}$$

\# pts " " right hand " "

$$\rho \frac{dp_k}{dt} dq_k = \rho \dot{p}_k dq_k$$

Total moving into dV as per picture

$$\rho(\dot{q}_k dp_k + \dot{p}_k dq_k)$$

\# moving <u>out</u> (edges at $p_k + dp_k$, $q_k + dq_k$)

$$\simeq \left(\rho \dot{q}_k + \frac{\partial}{\partial q_k}(\rho \dot{q}_k) dq_k\right) dp_k$$

$$+ \left(\rho \dot{p}_k + \frac{\partial}{\partial p_k}(\rho \dot{p}_k) dp_k\right) dq_k$$

Net increase inside is $-(\underline{\text{out}} - \underline{\text{in}})$

explicit ch. in ρ / time

$$\frac{\partial \rho}{\partial t} dq_k dp_k = -\left(\frac{\partial}{\partial q_k}(\rho \dot{q}_k) + \frac{\partial}{\partial p_k}(\rho \dot{p}_k)\right) dq_k dp_k$$

sum over all k

terms cancel from ⊛!

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial \rho}{\partial q_k}\dot{q}_k + \cancel{\rho \frac{\partial \dot{q}_k}{\partial q_k}} + \frac{\partial \rho}{\partial p_k}\dot{p}_k + \cancel{\rho\frac{\partial \dot{p}_k}{\partial p_k}} \right) = 0$$

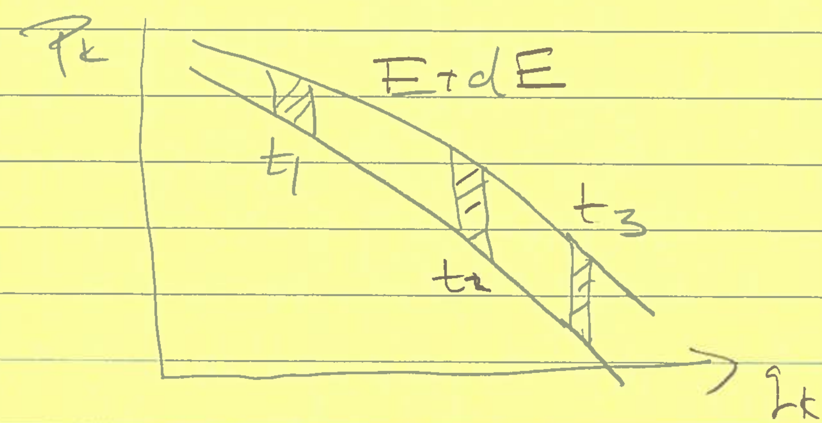
Q! where is the physics? Now!

⊛ Hamilton's eqns $\dot{q}_k = \frac{\partial H}{\partial p_k}$; $\dot{p}_k = -\frac{\partial H}{\partial q_k}$

$$\Rightarrow \frac{\partial \dot{q}_k}{\partial q_k} = \frac{\partial H}{\partial q_k \partial p_k} \quad \frac{\partial \dot{p}_k}{\partial p_k} = -\frac{\partial H}{\partial p_k \partial q_k}$$

$$\frac{\partial \rho}{\partial t} + \sum_k \left(\frac{\partial \rho}{\partial q_k}\dot{q}_k + \frac{\partial \rho}{\partial p_k}\dot{p}_k\right) = \boxed{\frac{d\rho}{dt} = 0}$$

Draw 2 hypersurfaces in 2n-dim space



$\Delta p_k, \Delta q_k$ must change to preserve ρ

Example: Particle beam

focus

$R' = \frac{1}{2} R$

$$V = (\pi p_0^2)(\pi R^2) = V' = (\pi p_0')^2 \left(\pi \left(\frac{R}{2}\right)^2\right)$$

$$\Rightarrow p_0' = 2 p_0$$