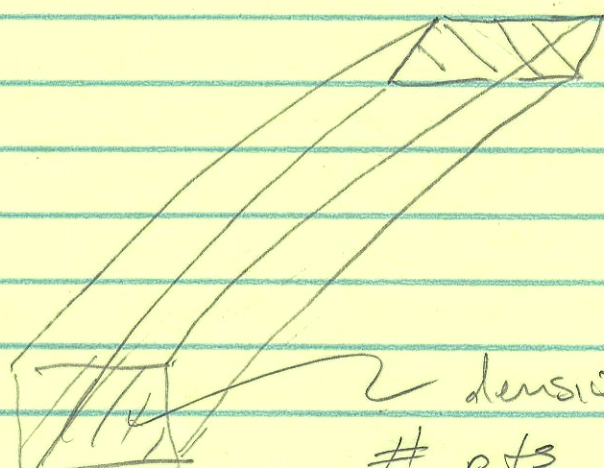
9/9/22 ⓪

Announcements

Reading: Ch. 8
HW2 Sept. 19, HW1 Solns posted
Quiz remarks

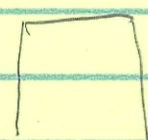
Last time Liouville's Theorem



density of points $\rho(p,q)$
\# pts $\rho\, dq\, dp$

$\rho = \frac{dN}{dq\,dp}$ Analogy to fluid: $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$

For inf. region



dp_k

dq_k

we calculated flux in − flux out

$$= \frac{\partial \rho}{\partial q_k}\dot{q}_k + \rho\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \rho}{\partial p_k}\dot{p}_k + \rho\frac{\partial \dot{p}_k}{\partial p_k}$$

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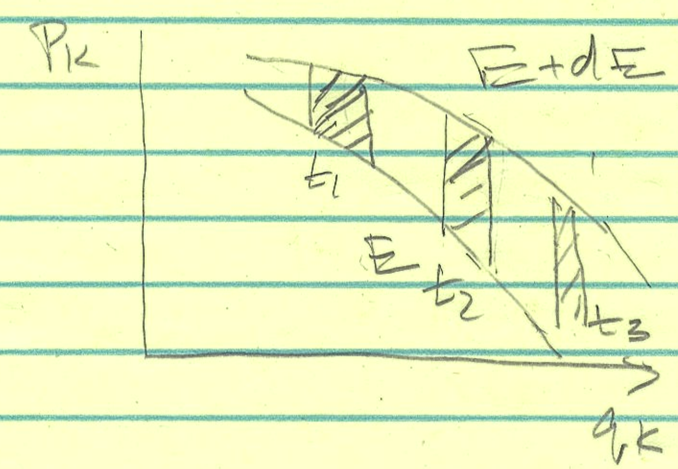
2 + 4 cancel since $\dot{q}_k = \frac{\partial H}{\partial p_k}$; $\dot{p}_k = \frac{-\partial H}{\partial q_k}$

$$\Rightarrow 2 + 4 = \frac{\partial H}{\partial q_k \partial p_k} - \frac{\partial H}{\partial p_k \partial q_k} = 0$$

So our "continuity eqn" is (sum over k)

$$\frac{\partial \rho}{\partial t} + \sum_k \left(\frac{\partial \rho}{\partial q_k} \dot{q}_k + \frac{\partial \rho}{\partial p_k} \dot{p}_k \right) = \frac{d\rho}{dt} = 0 !$$

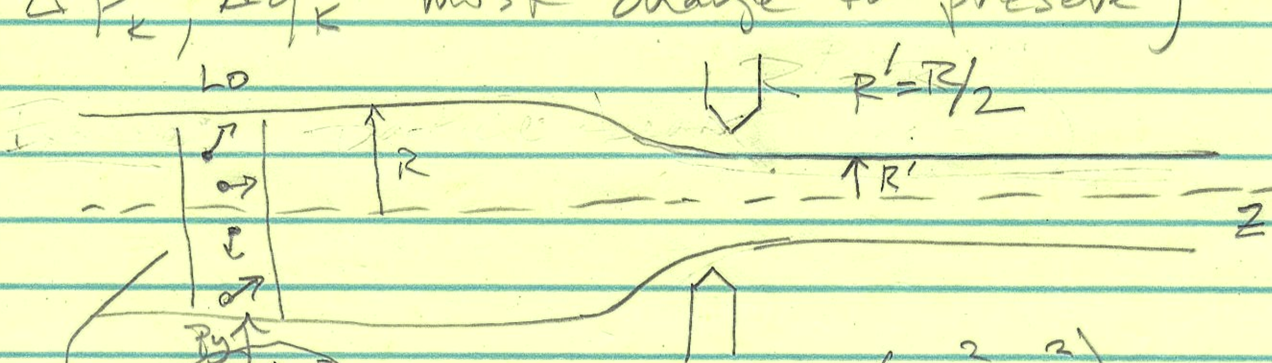
Draw 2 hypersurfaces in 2N-dim space



Δp_k, Δq_k must change to preserve ρ

Prob. 13.35

L_0 doesn't change

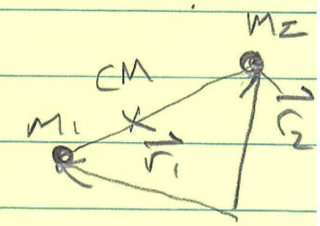


$$V = L_0 \left(\pi p_0^2 \pi R^2 \right)$$

$$\boxed{p_0' = 2p_0} = V' = L_0 \left(\pi p_0'^2 \pi (R/2)^2 \right)$$

Ch. 8 Orbits w central forces

3D 2-body problem + reduction to effective 1 body, 1D problem.



(A) CM + relative coords

CM position $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

total mass $M = m_1 + m_2$ rel. position $\vec{r} = \vec{r}_1 - \vec{r}_2$

Invert $\vec{r}_1 = \vec{R} + \dfrac{m_2}{M}\vec{r}$ $\quad \vec{r}_2 = \vec{R} - \dfrac{m_1}{M}\vec{r}$

KE $T = \frac{1}{2}m\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m\dot{\vec{r}}_2^{\,2}$

$$= \underbrace{\tfrac{1}{2}M\dot{\vec{R}}^{2}}_{\text{CM motion}} + \underbrace{\tfrac{1}{2}\frac{m_1 m_2}{M}\dot{\vec{r}}^{2}}_{\text{relative motion}}$$

$\dfrac{m_1 m_2}{M} = \mu$

reduced mass $\mu = \dfrac{m_1 m_2}{M}$

$$\mathcal{L} = T - U = \underbrace{\tfrac{1}{2}M\dot{\vec{R}}^2}_{\mathcal{L}_{CM}} + \underbrace{\left(\tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\right)}_{\mathcal{L}_{rel}}$$

Consider only those cases where $U(\vec{r}_1, \vec{r}_2) = U(r)$? Gravity, Coulomb, ...

$\vec{R} = (X, Y, Z)$ $\quad i = x, y, z \quad X_i = X, Y, Z$

CM motion $\frac{\partial \mathcal{L}_{cm}}{\partial X_i} = \frac{d}{dt}\frac{\partial \mathcal{L}_{cm}}{\partial \dot{X}_i} = M\ddot{X}_i = 0$

$\frac{d}{dt}(\underbrace{M\dot{X}_i}_{P_{cm}}) = 0$

$\Rightarrow P_{cm}$ conserved

$\vec{R}$ = "ignorable" $\quad \mathcal{L}$ depends only on $\dot{\vec{R}}$

$$\frac{\partial \mathcal{L}_{rel}}{\partial x_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$

$$-\frac{\partial}{\partial x_i} U = \frac{d}{dt}\frac{\partial \mathcal{L}_{rel}}{\partial \dot{x}_i} = \mu \ddot{x}_i$$

$\Rightarrow \hat{=}$ single particle w/ mass μ

Now we've reduced the problem to an effective one particle, 3D one. However, we can go further!

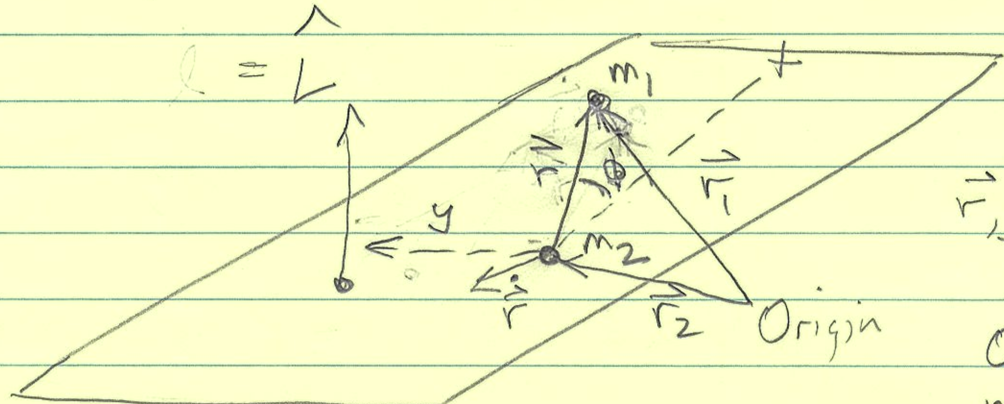
$$\vec{L} = \vec{r}_1 \times \vec{P}_1 + \vec{r}_2 \times \vec{P}_2$$

In CM frame $\vec{R} = 0$ $\quad \vec{r}_1 = \frac{m_2}{M}\vec{r}$ $\quad \vec{r}_2 = \frac{m_1}{M}\vec{r}$

$$\Rightarrow \boxed{\vec{L} = \frac{m_1 m_2}{M} \vec{r} \times \dot{\vec{r}} = (\vec{r} \times \mu \dot{\vec{r}})}$$

i.e. same form as our single particle, mass μ position $\vec{r}$

Since there is no external torque, $\vec{L}$ is conserved $\dot{\vec{L}} = 0$, in particular $\hat{L}$ is same throughout motion. This $\Rightarrow$ $\vec{r}$ $\dot{\vec{r}}$ are both $\perp$ $\vec{L}$, i.e. they always lie in a plane!



$\vec{r}, \vec{r}'$ in plane

Origin not nec.

Choose coordinates in this plane r, ϕ

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$$

$\boxed{\phi}$ $\mathcal{L}$ ind. of ϕ — ϕ is ignorable too!

A.M. cons.: $$\frac{\partial \mathcal{L}}{\partial \phi} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt}\underbrace{\left(\mu r^2 \dot{\phi}\right)}_{l = L_z}$$

check this!

$$\vec{r} = r(\cos\phi, \sin\phi)$$

$$\dot{\vec{r}} = \dot{r}(\cos\phi, \sin\phi) + r(-\sin\phi\,\dot{\phi}, \cos\phi\,\dot{\phi})$$

use $\vec{r} \times \dot{\vec{r}} = r^2\dot{\phi}$