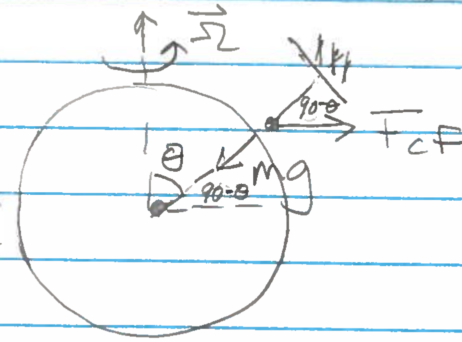
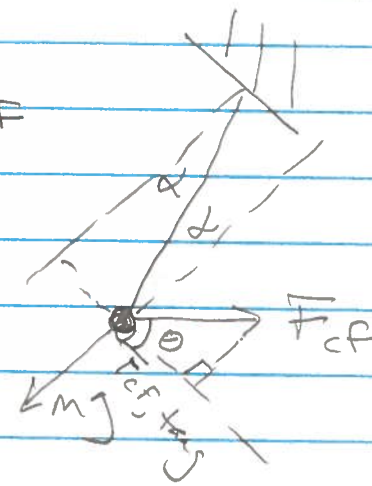
Exam 2 review 11/28/22 ①

<u>Ex. 1</u> 9.13 Show that angle between a plumb line and the direction of Earth's center is $\sin\alpha = R\Omega^2 \sin 2\theta / (2g)$





$$\vec{F}_{cf} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$
$$= m\Omega^2 R \sin\theta \, \hat{\rho}$$
$$\hat{\rho} = \sin\theta \, \hat{r} + \cos\theta \, \hat{\theta}$$

So $g_r = g_0 - \Omega^2 R \sin^2\theta$

$$g_{tang} = \Omega^2 R \sin\theta \cos\theta = \frac{1}{2}\Omega^2 R \sin 2\theta$$

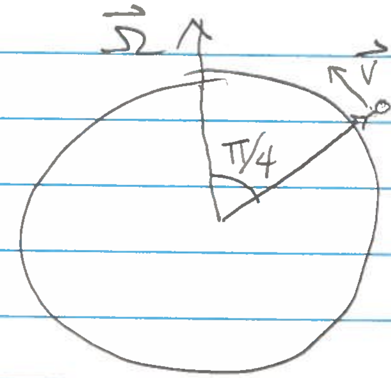
$$\tan\alpha = \frac{g_{tang}}{g_r} = \frac{\frac{1}{2}\Omega^2 R \sin 2\theta}{g_0 - \Omega^2 R \sin^2\theta} \simeq \frac{\Omega^2 R \sin 2\theta}{2 g_0}$$

<u>Ex. 2</u> A little league baseball pitcher can throw the ball at $20 ms^{-1}$ to the catcher 10 m away. The field is at latitude 45° and the ball is thrown north. What is the lateral deflection due to the Coriolis Force (magnitude + direction)

Ans $\vec{F}_{cor} = 2m\vec{v} \times \vec{\Omega}$ acts over $\frac{1}{2}$ sec $= \frac{10m}{20ms^{-1}}$ on a ball with no initial transverse speed – deflection is therefore $\frac{1}{2}at^2$

← we need to find $a = F_{cor}/m$

(but 1st, note that in the ½ s, ball falls 1.2 m, so pitcher needs to throw up a bit, but let's not worry)



$$F_{cor} = 2m\vec{v}\times\vec{\Omega}$$

$$= 2mv\Omega \sin 45^\circ$$

$$= \sqrt{2}\, m v \Omega$$

$$a = \frac{F_{cor}}{m} = \sqrt{2}\, v\Omega$$

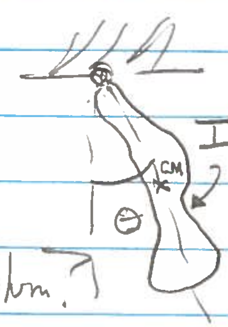
$$\text{deflection} \cong \frac{1}{2}\left(\sqrt{2}\, v\Omega\right)\left(\frac{1}{2}\text{ sec}\right)^2$$

$$= \frac{\sqrt{2}}{8}\left(\frac{20\text{ m}}{\text{s}}\right)\left(\frac{2\pi}{24\text{ hr}}\right)$$

$$= 0.177\left(20\,\frac{\text{m}}{\text{s}}\right)\left(7.2\times10^{-5}\,\frac{\text{rad}}{\text{s}}\right) = 0.0025\text{ m}$$

Direction is $\vec{v}\times\vec{\Omega}$ = into page (east)

Ex. 3 10.13 Compound pendulum
CM is distance a from pivot



a) what is freq. of small osc.?
b) Length of equivalent simple pendulum?
c) evaluate for rod

Ex. 3 cont'd

a) $L = I\dot{\theta}$; $\dot{L} = I\ddot{\theta} = \tau$ (torque)

$$= mga\sin\theta$$
$$\simeq mga\theta$$

$$I\ddot{\theta} \cong mga\theta, \quad \text{soln } A\sin(\omega t+\delta),$$

$$\omega = \sqrt{\frac{mga}{I}}$$

b) Compare simple pendulum $\omega = \sqrt{g/\ell}$

$$\Rightarrow \ell = \frac{I}{ma}$$

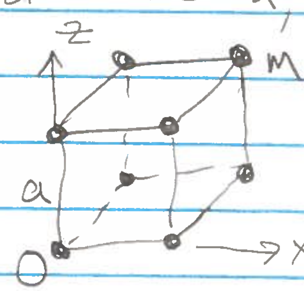
c) I of uniform rod around one end $I = \frac{m}{L}\int_0^L dx\, x^2$

$$= \frac{M}{L}\frac{L^3}{3} = \frac{1}{3}mL^2 \Rightarrow \omega = \sqrt{\frac{mga}{\frac{1}{3}mL^2}}$$

$$a = L/2 \Rightarrow \omega = \sqrt{\frac{3}{2}g/L}$$

Ex. 4 Find inertia tensor of set of 8 equal masses at corners of cube of side a, held together by massless rods

a) Find I_{ij} for axes along edges



for each sum points lie in right plane

$$I_{xx} = \sum_\alpha m_\alpha(y_\alpha^2+z_\alpha^2) = 4ma^2+4ma^2$$
$$= 8ma^2$$

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha = -2ma^2$$

only 2 have $x,y>0$

So $\quad I\Big|_{\text{around edge}} = ma^2 \begin{bmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{bmatrix}$

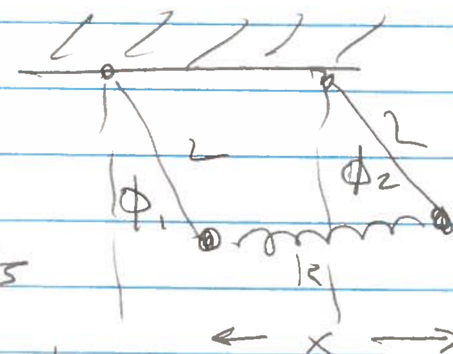
b) calculate $\underline{\underline{I}}$ for axis through center:

$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2) = 4ma^2$$

since every pt. has $y_\alpha, z_\alpha = \pm a/2$

$$I_{xy} = -\sum_\alpha m_\alpha (x_\alpha y_\alpha) = 0$$

since for every mass with $x_\alpha = +a/2$ there is another one (with same y_α) with $-a/2$



Ex. 5 Coupled pendula

Equil. length of spring is = to distance between pivots

Find a) Lagrangian + b) normal modes

$$T = \frac{1}{2} M L^2 (\dot{\phi}_1^2 + \dot{\phi}_2^2)$$

$$U_{spring} = \frac{1}{2} k x^2 = \frac{1}{2} k L^2 (\sin\phi_1 - \sin\phi_2)^2$$

$$\cong \frac{1}{2} k L^2 (\phi_1 - \phi_2)^2 \quad \text{(small angles)}$$

$$U_{grav} = mgL(1-\cos\phi_1) + mgL(1-\cos\phi_2)$$

$$\cong mg\frac{L}{2}(\phi_1^2 + \phi_2^2)$$

$$\Rightarrow \mathcal{L} \stackrel{\sim}{=} \underbrace{\tfrac{1}{2}mL^2(\dot\phi_1^2+\dot\phi_2^2) - \tfrac{1}{2}kL^2(\phi_1-\phi_2)^2 + \tfrac{1}{2}mgL(\phi_1^2+\phi_2^2)}$$

5

a) Lagrange's eqns:

$$/mL^2 \qquad \ddot\phi_1 = -\omega_0^2\phi_1 + \Omega_0^2(\phi_2-\phi_1)$$

$$\ddot\phi_2 = -\omega_0^2\phi_2 - \Omega_0^2(\phi_2-\phi_1)$$

where $\omega_0^2 = \frac{g}{l}$, $\Omega_0^2 = \frac{k}{m}$

b) Normal modes

$$\text{"}\underline{M}\text{"} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{"}\underline{K}\text{"} = \begin{bmatrix} \omega_0^2+\Omega_0^2 & -\Omega_0^2 \\ -\Omega_0^2 & \omega_0^2+\Omega_0^2 \end{bmatrix}$$

$$\det(\text{"}\underline{K}\text{"} - \text{"}\omega^2 M\text{"}) = \det \begin{bmatrix} \omega_0^2+\Omega_0^2-\omega^2 & -\Omega_0^2 \\ -\Omega_0^2 & \omega_0^2+\Omega_0^2-\omega^2 \end{bmatrix}$$

$$= (\omega_0^2+\Omega_0^2-\omega^2)^2 - \Omega_0^4$$

$$= \omega_0^4 + \omega^4 + 2\omega_0^2\Omega_0^2 - 2\omega^2\Omega_0^2 - 2\omega^2\omega_0^2 = 0$$

Factorize $(\omega_0^2-\omega^2)(\omega_0^2+2\Omega_0^2-\omega^2) = 0$

solns $\omega_1 = \omega_0 \qquad \omega_2 = \sqrt{\omega_0^2+2\Omega_0^2}$

substitute into "K" $-\omega^2$ "M"

ω_0 Eigenvalue 1 $\left.\begin{bmatrix} \; \end{bmatrix}\right|_{\omega_1} = \Omega_0^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow$ Eigenvector $\propto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ in-phase motion equal amplitudes

$\sqrt{\omega_0^2+2\Omega_0^2}$ Eigenvalue 2 $\left.\begin{bmatrix} \; \end{bmatrix}\right|_{\omega_2} = -\Omega_0^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

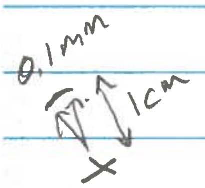
Ex. 5 cont'd

$\Rightarrow$ Eigenvector $\propto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ out of phase motion, equal amplitudes

Ex. 6 In Rutherford expt, $\frac{d\sigma}{d\Omega}$ for 6.5 MeV α's at $120°$ off a silver foil is 0.5 barns/sr. Total of 10^{10} α's hit the foil thickness $1\,\mu m$, and they are detected with counter of area $0.1\,mm^2$ 1 cm from target, what is N_{sc}?

use $\rho_{Ag} = 10.5 \times 10^5\ kg/m^3$

1st, calculate n_{tar}



$$n_{tar} = \frac{\rho t}{M_{Ag}} = \frac{(10.5 \times 10^5\,kg/m^3)(10^{-6}\,m)}{\underbrace{108}_{A}\,(1.66 \times 10^{-27}\,kg)}$$

$(1.66 \times 10^{-27}\,kg) \rightarrow$ mass/nucleon

$$= 5.86 \times 10^{22}\,m^{-2}$$

1 barn $= 10^{-28}\,m^2$

Solid angle subtended is $\Delta\Omega = \frac{(10^{-8}\,m^2) \cdot 4\pi}{(10^{-2}\,m)^2}$

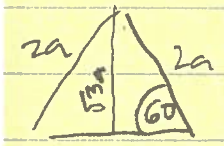
$$= 4\pi\, 10^{-4}$$

$$\underbrace{N_{sc} = N_{inc}\, n_{tar} \frac{d\sigma}{d\Omega} \Delta\Omega}_{\text{basic scattering eqn.}} = 10^{10} \cdot (5.86 \times 10^{22}\,m^{-2})(0.5 \times 10^{-28})$$

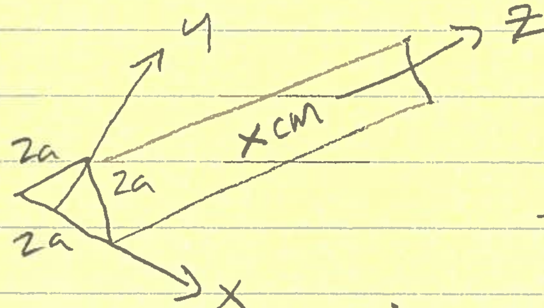
$$\times 4\pi \times 10^{-4} = \underline{\underline{36.8\ \alpha}}$$

Ex. 7 Problem 10.28 from book

Find $\underline{\underline{I}}$ for triangular prism through cm



$\sin 60^\circ = \frac{\sqrt{3}}{2}$



Area of eq. triangle is

$\frac{1}{2} b \cdot h =$

$\frac{1}{2} \cdot 2a \cdot \sqrt{3}a = \sqrt{3}a^2$

$\Rightarrow$ mass density is $\frac{M}{\sqrt{3}a^2 L}$

$$\underline{I}_{zz} = \frac{M}{\sqrt{3}a^2 L} \int_{\text{limits}} dx\, dy \int_{-L/2}^{L/2} dz\, (x^2 + y^2)$$ around base center

$$= \frac{M}{\sqrt{3}a^2} \int_{\text{limits}} dx\, dy\, (x^2 + y^2)$$

$$= \frac{M}{\sqrt{3}a^2} \left[\int dx\, x^2 \int dy + \int dx \int dy\, y^2 \right]_{\text{limits}}$$

$$= \frac{M}{\sqrt{3}a^2} \Bigg[\underbrace{\int_{-a}^{0} dx\, x^2 \int_0^{(a+x)\sqrt{3}} dy}_{A} + \underbrace{\int_0^a dx\, x^2 \int_0^{(a-x)\sqrt{3}} dy}_{B}$$

$$+ \underbrace{\int_{-a}^{0} dx \int_0^{(a+x)\sqrt{3}} y^2\, dy}_{C} + \underbrace{\int_0^a dx \int_0^{(a-x)\sqrt{3}} dy\, y^2}_{D} \Bigg]$$

symmetry $A = B = \int_0^a dx\, x^2 (a-x)\sqrt{3} = \left(\frac{a x^3}{3} - \frac{a x^4}{4} \Big|_0^a \right) \sqrt{3}$

$= \frac{\sqrt{3}a^4}{12}$

$$C = D = \int_0^a dx \int_0^{(a-x)\sqrt{3}} dy\, y^2 = \int_0^a dx\, (a-x)^3 \sqrt{3}$$

$$= \frac{\sqrt{3}\, a^4}{4}$$

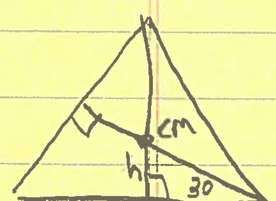
$$\Rightarrow \quad A + B + C + D = \frac{2\cdot\sqrt{3}}{12} a^4 + \frac{2\cdot\sqrt{3}}{4} a^4$$

$$= \frac{2\sqrt{3}}{3} a^4$$

$$\Rightarrow \quad I_{zz} \text{ around base center} = \frac{2Ma^2}{3}$$

To find I_{zz}^{cm} use // axis thm:

$$I_{zz}^{base} = I_{zz}^{cm} + Mh^2$$



$$\frac{2}{3} Ma^2 = I_{zz}^{cm} + M\left(\frac{a}{\sqrt{3}}\right)^2$$

$$I_{zz}^{cm} = \left(\frac{2}{3} - \frac{1}{3}\right) Ma^2 = \frac{Ma^2}{3}$$

$$\tan 30 = \frac{h}{a} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow h = a/\sqrt{3}$$

$I_{xz} = I_{yz} = 0$ reflection in xy

$I_{xy} = 0$ reflection in yz

$$\underline{I_{xx}} = \frac{M}{\sqrt{3}a^2 L} \int dx\,dy\,dz\,(y^2 + z^2)$$

$$= \frac{M}{\sqrt{3}a^2 L}\left[\underbrace{\overset{A}{\int dx\,dy\,dz\; y^2}}_{\frac{1}{2} I_{zz}} + \overset{B}{\int dx\,dy\,dz\; z^2}\right]$$

$$B = \int_{\text{limits}} dx\,dy \int_{-L/2}^{L/2} dz\; z^2 = A_{\text{triangle}} \cdot \frac{L^3}{12}$$

10.28 says "height" is h — I think This means length of prism L.

$$A_{\text{triangle}} = \sqrt{3}a^2$$

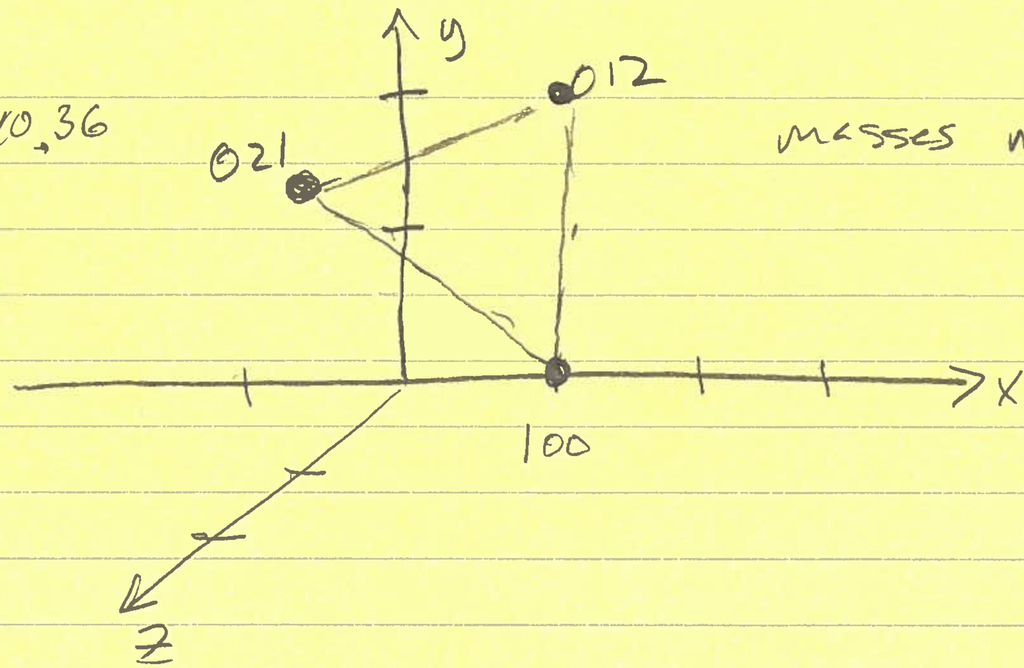
$$I_{xx} = \underbrace{\frac{Ma^2}{6}}_{A} + \frac{M}{\cancel{\sqrt{3}a^2}\cancel{L}} \cdot \frac{\cancel{\sqrt{3}a^2}\, L^{\cancel{3}\,2}}{12} = \frac{M}{12}\left(2a^2 + L^2\right)$$

$I_{yy} = I_{xx}$ by symmetry,

$$\underline{\underline{I}} = \frac{Ma^2}{12}\begin{bmatrix} 2a^2 + L^2 & 0 & 0 \\ 0 & 2a^2 + L^2 & 0 \\ 0 & 0 & 4a^2 \end{bmatrix}$$

Ex. 8

Problem 10.36



masses m =

Construct $\underline{\underline{I}}$:

$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2) = ma^2(0+5+5) = 10ma^2$$

$$I_{yy} = \sum_\alpha m_\alpha (x_\alpha^2 + z_\alpha^2) = ma^2(1+4+1) = 6ma^2$$

$$I_{zz} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = ma^2(1+1+4) = 6ma^2$$

$$I_{xy} = -\sum_\alpha m_\alpha (x_\alpha y_\alpha) = 0$$

$$I_{xz} = -\sum_\alpha m_\alpha (x_\alpha z_\alpha) = 0$$

$$I_{yz} = -\sum_\alpha m_\alpha (y_\alpha z_\alpha) = -ma^2(2+2) = -4ma^2$$

$$\underline{\underline{I}} = ma^2 \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{bmatrix}$$

E'value problem

$$\underline{\underline{I}} - \lambda \mathbb{1} = 0$$

$$0 = \det\left(\underline{\underline{I}} - \lambda \mathbb{1}\right)$$

$$= (10-\lambda)\left[(6-\lambda)^2 - 16\right] \quad \text{units of } ma^2$$

$$= (10-\lambda)^2 (2-\lambda)$$

2 principal moments, $2ma^2$ and $10ma^2$

$10\ ma^2$ has 2 principal axes
$2\ ma^2$ has 1 " "

Find e'vectors:

$\underline{\lambda = 10}$ $\quad \underline{\underline{I}} - \lambda \underline{\mathbb{1}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & -4 \\ 0 & -4 & -4 \end{bmatrix}$

solve $(\underline{\underline{I}} - \lambda \underline{\mathbb{1}})\vec{a} = 0$

$$\vec{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{a}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$ ← not normalized

$\underline{\lambda = 2}$ $\quad \underline{\underline{I}} - \lambda \mathbb{1} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \end{bmatrix}$

solve

$$\vec{a}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$