Catenary

The *catenary* curve (from the Latin for "chain") is the shape assumed by a uniform chain of fixed length supported at its ends under the influence of gravity. Let the curve be described by a function $y(x)$ with endpoints $y(x_1) = y_1$, $y(x_2) = y_2$. Let μ be the (constant) mass per unit length. The shape is then found by minimizing the gravitational potential energy,

$$E[y(x)] = \sum mgh = \int \mu g\, y(x)\, ds \int \mu g y \sqrt{1 + y'^2}\, dx,$$

while fixing the length

$$s[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx = \ell.$$

(A functional $J[y(x)]$ is an operation that takes as its argument the function $y(x)$ and yields a number: energy, length, "action," ....) Impose the constraint with a Lagrange multiplier and extremize the action

$$J = E + \lambda(s - \ell) = \int (\lambda + \mu g y)\sqrt{1 + y'^2}\, dx - \lambda\ell.$$

As in Section 6.5 in the text, varying $y(x)$ leads to the Euler equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = \mu g\sqrt{1 + y'^2} - \frac{d}{dx}\left[\frac{(\lambda + \mu g y)y'}{\sqrt{1 + y'^2}}\right]$$

$$= \frac{\mu g(1 + y'^2)^2}{(\sqrt{1 + y'^2})^3} - \frac{\lambda y'' + \mu g(yy'' + y'^2 + y'^4)}{(\sqrt{1 + y'^2})^3} = \frac{\mu g(1 + y'^2 - yy'') - \lambda y''}{(\sqrt{1 + y'^2})^3} = 0.$$

The expression is simpler than it might have been because the derivative

$$\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = \frac{y''}{(\sqrt{1 + y'^2})^3}$$

comes out nice (for exponents other than 2 inside the square root there are more terms), and because y'^4 terms cancel between the $\partial f/\partial y$ and $\partial f/\partial y'$ terms. Variation of λ leads to

$$\frac{\partial J}{\partial \lambda} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx - \ell = 0.$$

Thus, after all this effort we are led to a deceptively simple-looking differential equation plus an integral constraint,

$$(\lambda/\mu g + y)\, y'' = 1 + y'^2, \qquad \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx = \ell. \qquad (*)$$

which we must solve for given boundary conditions.

The solution makes use of properties of the hyperbolic functions

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$

and in particular the properties

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \frac{d}{dx}\sinh x = \cosh x, \qquad \cosh^2 x = 1 + \sinh^2 x.$$

Thus, the function $y = \cosh x$ satisfies $y\,y'' = 1 + y'^2$, This is not yet the solution, because we still have to account for boundary conditions and the constraint. First note that $y\,y'' = 1 + y'^2$ remains true after a scaling, $y = a\,\cosh(x/a)$. This is good because x actually has units, while the argument of cosh must be dimensionless; and also because it will allow us to satisfy the length constraint. We can also translate the minimum anywhere we need by shifting x to $x - b$. Finally, we must address the $\lambda/\mu g$ term, but that can be done by adding a constant to y. So, with all of this we have the general shape of the curve

$$y(x) = a\,\cosh\Big(\frac{x-b}{a}\Big) + c, \qquad y' = \sinh\Big(\frac{x-b}{a}\Big), \qquad y'' = \frac{1}{a}\,\cosh\Big(\frac{x-b}{a}\Big).$$

This function satisfies the differential equation for any values of a, b, and c, as long as $\lambda/\mu g + c = 0$, which determines λ. The value of the coefficients a, b, c are determined by the length constraint,

$$\begin{aligned} s &= \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx = \int_{x_1}^{x_2} \sqrt{1 + \sinh^2\Big(\frac{x-b}{a}\Big)}\,dx \\ &= \int_{x_1}^{x_2} \cosh\Big(\frac{x-b}{a}\Big)\,dx = a\left[\sinh\Big(\frac{x_2-b}{a}\Big) - \sinh\Big(\frac{x_1-b}{a}\Big)\right] = \ell. \end{aligned}$$

plus boundary conditions.

As an example, let $x_1 = -\frac{1}{2}d$ and $x_2 = \frac{1}{2}d$, with $y_1 = y_2 = h$. Symmetry in $\pm x$ says $b = 0$. Then the length constraint says

$$\frac{\sinh(d/2a)}{(d/2a)} = \frac{\ell}{d},$$

which, since $\sinh x > x$, has a solution for any $\ell \geq d$ and serves to determine a; and the value $y = h$ at $x = \pm d$ fixes c.

The figure (following page) shows results for chains of various lengths. The longest one falls below $y = 0$.

Catenary Curves