Euler Angle Rotations

Here is the quick summary of all there is to know about Euler angles (note that the signs of the sines are opposite those in the text).

The orientation of a rigid body requires the specification of three quantities that can be taken to be the angles in the product of three rotations R_ϕ, R_θ, R_ψ, where as in (11.91), (11.93), and (11.95) in the text

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Starting from a given vector in space, a general orientation can be obtained from a rotation by ϕ about the z-axis; followed by a rotation by θ about the new x-axis (often called the "line of nodes"), followed by a rotation by ψ about the newer z-axis, so that $\boldsymbol{x} \to R\boldsymbol{x}$, where

$$R = R_\psi R_\theta R_\phi = \begin{pmatrix} \cos\psi\,\cos\phi - \cos\theta\,\sin\phi\,\sin\psi & -\cos\psi\,\sin\phi - \cos\theta\,\cos\phi\,\sin\psi & \sin\psi\,\sin\theta \\ \sin\psi\,\cos\phi + \cos\theta\,\sin\phi\,\cos\psi & -\sin\psi\,\sin\phi + \cos\theta\,\cos\phi\,\cos\psi & -\cos\psi\,\sin\theta \\ \sin\theta\,\sin\phi & \sin\theta\,\cos\phi & \cos\theta \end{pmatrix}.$$

Columns of R are presented in the text in (11.99).

The rotated position of a point initially at $\boldsymbol{x}_0$ is $\boldsymbol{x} = R\boldsymbol{x}_0$, and its velocity is $\dot{\boldsymbol{x}} = \dot{R}\boldsymbol{x}_0 = \dot{R}R^{-1}\boldsymbol{x}$, where

$$\dot{R} = \dot{\psi}\, R'_\psi R_\theta R_\phi + \dot{\theta}\, R_\psi R'_\theta R_\phi + \dot{\phi}\, R_\psi R_\theta R'_\phi$$

(prime $'$ denotes the derivative of the rotation with respect to its angle). On the other hand, any product of rotations can be written as a single rotation by an angle (call it Θ) about an axis (call it $\hat{\boldsymbol{n}}$), and the velocity can be written as $\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x}$, where $\boldsymbol{\omega}$ is a vector with magnitude $\dot{\Theta}$ in the direction $\hat{\boldsymbol{n}}$. In the text the components of $\boldsymbol{\omega}$ in the body frame (final frame) are given in (11.102),

$$\boldsymbol{\omega} = (\dot{\phi}\,\sin\theta\,\sin\psi + \dot{\theta}\,\cos\psi)\,\hat{\boldsymbol{e}}_1 + (-\dot{\phi}\,\sin\theta\,\cos\psi + \dot{\theta}\,\sin\psi)\,\hat{\boldsymbol{e}}_2 + (\dot{\phi}\,\cos\theta + \dot{\psi})\,\hat{\boldsymbol{e}}_3,$$

so, $\omega^2 = \dot{\Theta}^2 = \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta$. A prodigious amount of algebra confirms that

$$\begin{aligned} \dot{R}R^{-1}\boldsymbol{x} = {} & [(\dot{\theta}\,\sin\psi - \dot{\phi}\,\sin\theta\,\cos\psi)\,z - (\dot{\psi} + \dot{\phi}\,\cos\theta)\,y]\,\hat{\boldsymbol{e}}_1 \\ & + [(\dot{\psi} + \dot{\phi}\,\cos\theta)\,x - (\dot{\theta}\,\cos\psi + \dot{\phi}\,\sin\theta\,\sin\psi)\,z]\,\hat{\boldsymbol{e}}_2 \\ & + [(\dot{\theta}\,\cos\psi + \dot{\phi}\,\sin\theta\,\sin\psi)\,y - (\dot{\theta}\,\sin\psi - \dot{\phi}\,\sin\theta\,\cos\psi)\,x]\,\hat{\boldsymbol{e}}_3 = \boldsymbol{\omega} \times \boldsymbol{x}. \end{aligned}$$