Some Classical Mechanics Prelim Exam Problems

1. A particle moves in a circular orbit under the influence of an attractive central force of magnitude $K\, e^{-r/a}/r^2$. The particle is slightly perturbed from the original orbit but continues to move in the same plane. Find the conditions on the original orbit so that the perturbed orbit is a stable oscillation about the original.

2. A particle is constrained to move on a sphere of radius a under the influence of a uniform gravitational field. Initially the particle moves horizontally with velocity v.

(a) Show that the particle either never falls below or never rises above its original height.

(b) Find the conditions such that the particle never falls below its original height.

3. Gas and debris of total mass m surround a star of mass M and initially are *not* confined to one plane. The radius of the star is negligible in comparison with the distances to the surrounding particles. Assume that $m \ll M$, so that one may neglect the gravitational interactions among the particles of gas and debris in comparison with the gravitational force exerted on the particles by the star. Due to internal friction, the cloud of material continually loses mechanical energy. Show that there is a maximum energy ΔE that can be lost in this way and that when this energy has been lost, the material must all lie in one circular ring around the star. In terms of the initial conditions of the gas and debris, find ΔE, the orbital radius of the ring, and the plane in which the ring lies.

4. A cannon is located on the surface of the earth at latitude λ in the northern hemisphere. A projectile is fired from the cannon with an initial velocity which has magnitude v_0, angle of inclination to the horizontal α, and horizontal component pointing directly east. Where (with respect to the cannon) does the projectile strike the earth? The distance travelled by and the height reached by the projectile are small compared with the radius of the earth. Compute your answer to first order in the angular velocity of the earth.

5. Particles of mass m and energy E are projected at a stationary target of particles of the same mass m. The potential between incident and target particles is

$$V(\boldsymbol{r}) = \begin{cases} -|V_0| & \text{for } |\boldsymbol{r}| < a, \\ 0 & \text{for } |\boldsymbol{r}| > a, \end{cases}$$

where $\boldsymbol{r}$ is the relative position of the centers of the particles. Find the laboratory differential cross section as a function of the laboratory scattering angle. Neglect relativistic and quantum effects.

6. Three simple pendulums of mass m and length ℓ are suspended from a ceiling. The points of suspension from the ceiling form a straight line, and the distance between adjacent suspension points is d. The middle mass is connected to each of the other two by a spring of equilibrium length d and spring constant k. Find the fundamental frequencies for small oscillations about the equilibrium of the system and describe the normal modes corresponding to each fundamental frequency. (Motion is *not* restricted to a plane.)

7. A mass m_1 moves on a frictionless horizontal table. The mass is connected by a string of length ℓ through a hole in the table to a mass m_2 that hangs vertically beneath the table in a constant gravitational field.

(*a*) Given the initial position $\boldsymbol{r}_0$ and velocity $\boldsymbol{v}_0$ in the plane of the table, find an equation that determines the maximum and minimum radii of the orbit of m_1 on the table. (Don't try to solve it.) For what initial conditions is the orbit circular?

(*b*) Find the frequency of oscillation of small perturbations from a circular orbit.

8. A small block of mass m moves around the rim of a Petri dish of radius R. The bottom of the dish is frictionless and the outer edge has coefficient of kinetic friction μ. At time $t = 0$ the block is at $\theta = 0$ and has speed $v = v_0$. Find $\theta(t)$, and in particular, how far the block moves.

9. A particle of rest mass m and charge $-e$ moves towards a fixed charge $+e$. The incident particle has kinetic energy E at infinity and angular momentum L about the fixed charge. Ignore radiation effects.

(*a*) In Newtonian mechanics, what is the distance of closest approach?

(*b*) In special relativity, write an expression for the distance of closest approach.

(*c*) In special relativity, find the angular momentum L such that the charges collide.

10. A solid cylinder of radius r and mass m (hence, moment of inertia $I = \frac{1}{2}mr^2$) rolls without slipping inside a larger hollow cylinder with inner radius R. The axes of the two cylinders are horizontal and parallel, and the motion takes place in the uniform gravitational field at the surface of the earth. The outer cylinder rotates about its fixed axis with constant angular *acceleration* α. In terms of the angle ψ, the displacement of the inner cylinder from its position for $\alpha = 0$, find the equation of motion for the inner cylinder. Find ψ_0, the displacement angle at which the inner cylinder may remain at rest. When does such a ψ_0 fail to exist?

11. It may be possible to detect gravitational radiation by exciting vibrational modes of a massive cylinder. The amplitude of motion in such a detector is expected to be small, and thus difficult to observe. Excitation of the detector, however, might indirectly be measured by connecting it to a less massive system, resonant at the same frequency. Here is a model for such an apparatus:

Consider a one-dimensional system of two masses and two springs with the same resonant frequency ω_0. The first spring, with $k_1 = M\omega_0^2$, has one end fixed and the other end connected to mass $m_1 = M$; the second spring, with $k_2 = m\omega_0^2$, connects m_1 and $m_2 = m$.

(a) What are the fundamental frequencies of the system to lowest order in m/M?

(b) If the system is excited by applying an impulse I_0 to M, discuss the energy distribution between the large spring/large mass and the small spring/small mass as a function of time. What is the maximum stretch of the small spring, and how long does it take to reach this condition after I_0 is applied? (Remember to keep only lowest order in m/M.)

12. A particle moves in a central force field,

$$\boldsymbol{F} = -\frac{k}{r^{\alpha+1}}\,\hat{\boldsymbol{r}} \qquad (\alpha > 0).$$

(a) Find the radius of circular orbits as a function of the angular momentum ℓ. Describe quantitatively the dependence of r on ℓ for possible values of α.

(b) If a small impulse in the radial direction is applied to the particle (thus leaving the angular momentum ℓ unaltered), for what values of α is the new orbit close to the old (that is, for what values of α are the circular orbits stable against such perturbations)?

(c) For the allowed α of part (b), find the frequency of oscillation under small perturbations.

13. Four particles confined to move on a circle of radius R are attached by springs stretched along the circumference, such that in equilibrium they are spaced uniformly. All springs are identical with spring constant K; three of the particles have mass M_A and the fourth has mass M_B. Find the frequencies and normal modes of vibration.

14. A particle with charge q, mass m, moves in a plane under the action of a force $\boldsymbol{F} = +a\boldsymbol{r}$, $a > 0$ and in the presence of a magnetic field $\boldsymbol{B}$ perpendicular to the plane. Under what conditions will the particle escape to infinity?

15. Consider a stretched string tied at both ends which undergoes small vibrations in one dimension perpendicular to the length of the string. Show that the Lagrangian is

$$L = \int_0^a \left[\frac{1}{2}\mu \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} T \left(\frac{\partial \phi}{\partial x} \right)^2 \right] dx,$$

where a is the length, μ is the linear mass density along the string, T is the tension, and $\phi(x,t)$ is the amplitude of the motion. Derive the equation of motion.