PHY4324–Electromagnetism II

Fall 2011

Test 1 – 55 minutes

Sept. 26, 2011

No other materials except calculators allowed. If you can't do one part of a problem, solve subsequent parts in terms of unknown answer–define clearly. **Do 3 of the 4 problems, CLEARLY indicating which you want graded by circling the problem number!**. *Each problem is worth 10 pts., for maximum of 30 points.*

1. Consider a superposition of two linear polarized waves traveling in the z direction with electric field

$$\mathbf{E}(z,t) = \mathrm{Re}\left[\hat{x}E_{10}e^{i(kz-\omega t)} + \hat{y}E_{20}e^{i(kz-\omega t)}\right], \tag{1}$$

where $E_{10} = E_1$ is real but $E_{20} = E_2 e^{i\phi}$, where E_1 and E_2 are real and positive, and the real number ϕ is the *phase difference* between the two components.

(a) (2 pts.) For $\phi = 0$, the resulting wave is still linearly polarized. What is the amplitude of the total electric field, and what is the direction of the polarization vector?

(b) (4 pts.) For $\phi = \pi/2$ and $E_1 = E_2$, the resulting wave is *circularly polarized.* What is the magnitude of the electric field as a function of time t at the point $z = 0$? In which direction does the electric field rotate (clock- or counterclockwise) as one views the wave coming towards the observer along the z-axis (i.e., you are sitting way down the positive z axis looking in the $-z$ direction).

(c) (4 pts.) For $\phi = \pi/2$ and $E_1 \neq E_2$, the wave is *elliptically polarized.* Determine the electric field at $z = 0$ as a function of time.

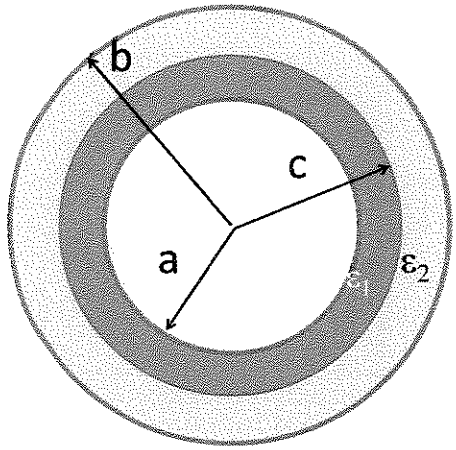
2. A long solenoid of height X is made of thin wire (diameter d) wrapped tightly around a cylinder of radius a. The density of conduction electrons in the wire is n, and the mean free time between collisions is τ_i.

   (a) (4 pts.) In terms of the quantities given, the electron charge and mass e and m, respectively, find the conductivity σ and the resistance R of the coil (Hint: find the total length of the wire).

(b) (4 pts.) In terms of the conductivity σ and/or R (no need to substitute from part (a)), what is the self-inductance of the coil?

(c) (2 pts.) At $t = 0$, the solenoid is connected to a battery. Find the characteristic delay time after which the current in the solenoid approaches its steady-state value.

3. A t-dependent voltage $V(t) = V_0 \cos(\omega t)$ is applied to a capacitor, which consists of two concentric conducting spheres of radii a and b ($a < b$). The space in



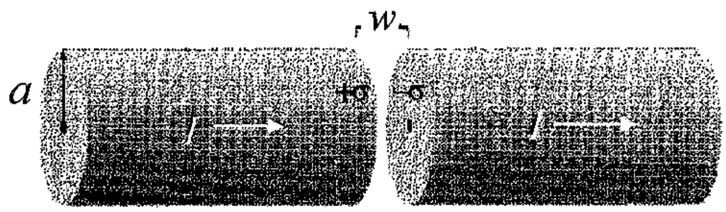
between the spheres is filled with two spherical shells made of *different* insulators, so that

$$\epsilon = \begin{cases} \epsilon_1 \ \text{for } a < r < c \\ \epsilon_2 \ \text{for } c < r < b. \end{cases}$$

(a) (4 pts.) Find the capacitance C.

(b) (4 pts.) Find the displacement current (direction and magnitude) in terms of C.

(c) (2 pts.) Find the magnetic field produced by the displacement current.



4. A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire of width $w \ll a$, forms a parallel plate capacitor.

   (a) (4 pts.) Find $\mathbf{E}$ and $\mathbf{B}$ in the gap as functions of the distance s from the axis and of the time t (assume charge density σ is zero at $t = 0$).

(b) (4 pts.) Find the energy density u_{em} and the Poynting vector $\mathbf{S}$ in the gap. What are the direction and magnitude of $\mathbf{S}$?

(c) (2 pts.) Verify the conservation of energy in the gap locally. (Hint: you may need $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{z}$.)

Solns test 1
Phy 4324 2011

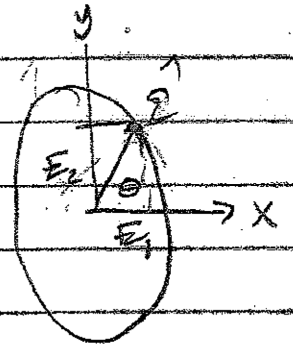
1) wave $\vec{E} = \mathrm{Re}\left[\hat{x} E_{10} e^{i(kz-\omega t)} + \hat{y} E_{20} e^{i(kz-\omega t)}\right]$

$E_{20} = E_2 e^{i\phi}$

a) For $\phi = 0$ $\quad \vec{E} = \left(E_1 \hat{x} + E_2 \hat{y}\right) \cos(kz - \omega t)$

Amplitude $\sqrt{\vec{E} \cdot \vec{E}} = \sqrt{E_1^2 + E_2^2}$

polarization $\hat{\varepsilon} = \dfrac{E_1 \hat{x} + E_2 \hat{y}}{\sqrt{E_1^2 + E_2^2}}$



$\theta = \tan^{-1} \dfrac{E_2}{E_1}$

independent of t !

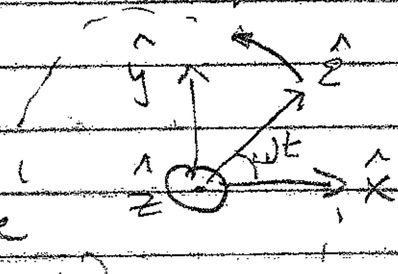
b) $z = 0, \quad \phi = \pi/2 \quad E_1 = E_2$

$\vec{E} = \mathrm{Re}\left[\left(E_1 \hat{x} + i E_1 \hat{y}\right) e^{i(kz-\omega t)}\right]$

$= E_1 \left(\hat{x} \cos \omega t + \hat{y} \sin \omega t\right)$

Amplitude $= \sqrt{E_1^2 \cos^2 \omega t + E_1^2 \sin^2 \omega t}$

$= E_1$



For $t \geq 0$
$\hat{\varepsilon}$ rotates counterclockwise
(left circularly polarized)

c) $z = 0 \quad \phi = \pi/2$

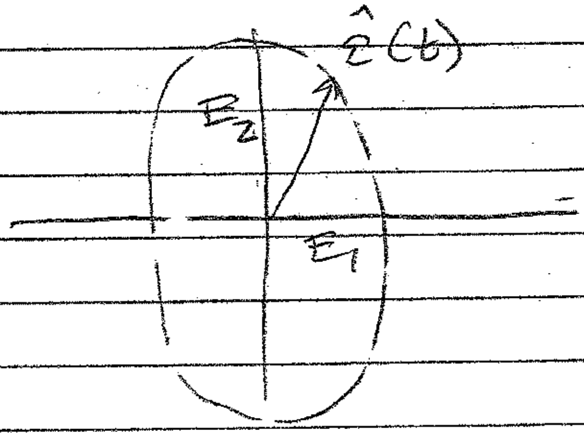
$$\vec{E} = \mathrm{Re}\left[(E_1\hat{x} + iE_2\hat{y})\, e^{i(\quad)}\right]$$

$$= E_1\hat{x}\cos\omega t + E_2\hat{y}\sin\omega t$$

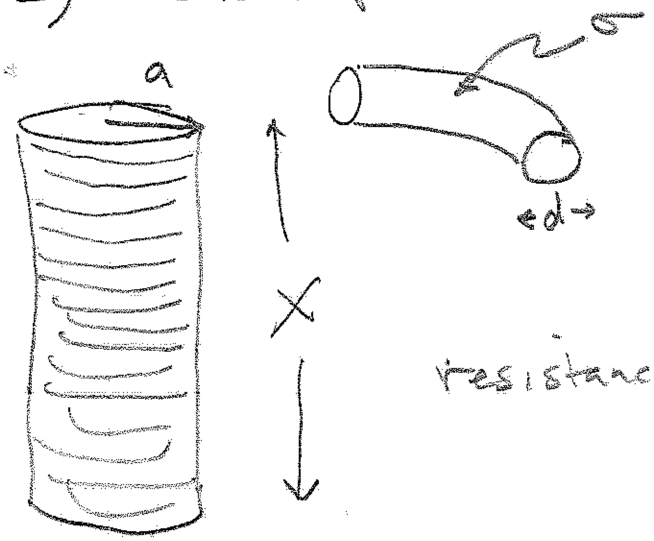
Amplitude $= \sqrt{E_1^2\cos^2\omega t + E_2^2\sin^2\omega t}$

not a constant!

angle $\tan^{-1}\dfrac{E_2\sin\omega t}{E_1\cos\omega t}$



2) Solenoid (a)



Tightly packed solenoid

$\Rightarrow$ length of wire

$$= l = 2\pi a X / d$$

resistance $R = \rho \frac{l}{A} = \frac{1}{\sigma} \frac{\left(\frac{2\pi a X}{d}\right)}{\pi d^2/4}$

$$\boxed{R = 8\frac{aX}{\sigma d^3} = \frac{8aXm}{ne^2\tau_i d^3}}$$

Drude conductivity

$$\boxed{\sigma = \frac{ne^2\tau_i}{m}}$$

(b) $B = \mu_0 N I$ where $N = \frac{1}{d}$ turns / length

Energy stored is

$$W = \frac{1}{2\mu_0}\int B^2 dz = \frac{\mu_0}{2} N^2 (\pi a^2 X) I^2 \equiv \frac{1}{2} L I^2$$

$$\Rightarrow \boxed{L = \mu_0 N^2 (\pi a^2 X) = \mu_0 \frac{\pi a^2 X}{d^2}}$$

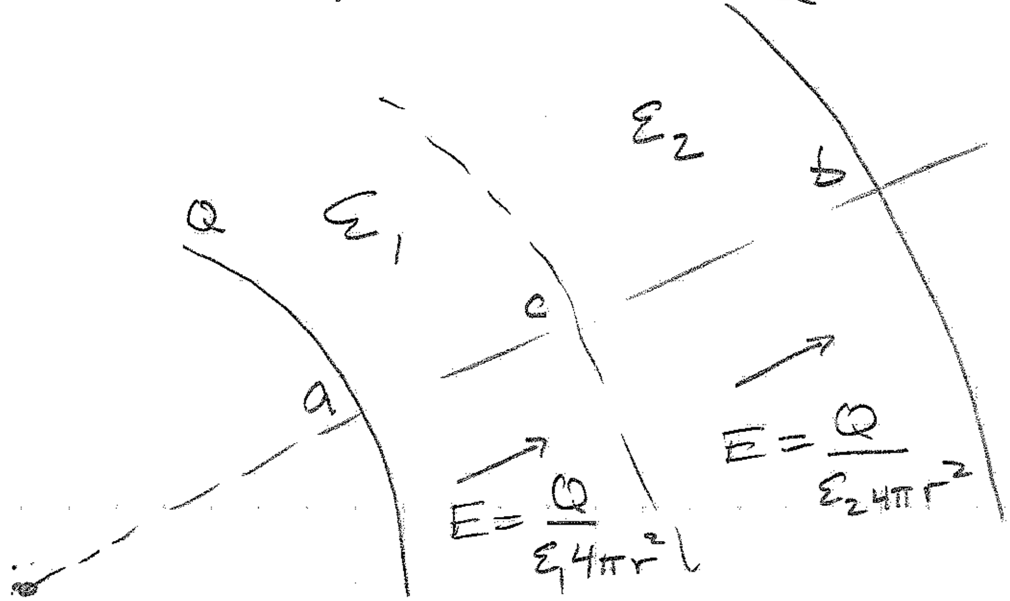
(c) Delay time for an inductor

$$\tau = L/R = \frac{\mu_0 \pi a^2 X / d^2}{8aX/(\sigma d^3)}$$

$$\boxed{\tau = \frac{\pi}{8}\mu_0 \sigma a d = \frac{\pi}{8m}\mu_0 n e^2 \tau_i a d}$$

3) (a) Capacitance: put Q on inner conductor

Electric fields are



$$V = V(a) - V(b) = -\int_b^a \vec{E}\cdot d\vec{r} = \int_a^b E\,dr$$

$$= \frac{Q}{4\pi\varepsilon_1}\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{Q}{4\pi\varepsilon_2}\left(\frac{1}{c} - \frac{1}{b}\right)$$

$$Q = CV \quad \Rightarrow \quad \boxed{C = \frac{1}{\frac{1}{4\pi\varepsilon_1}\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{1}{4\pi\varepsilon_2}\left(\frac{1}{c} - \frac{1}{b}\right)}}$$

(b) Displacement current density $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$

Displacement current $I_d = \int \vec{J}_d \cdot d\vec{a} = J_d 4\pi r^2$

$\vec{D}$ is displacement field due to free charges:

$$\vec{D} = \frac{Q}{4\pi r^2} \quad \Rightarrow \quad J_d = \frac{\dot{Q}}{4\pi r^2} = \frac{C}{4\pi r^2}\dot{V}$$

$$\Rightarrow \quad J_d = -\frac{C}{4\pi r^2} V_0 \omega \sin \omega t$$

$$\boxed{I_d = -C V_0 \omega \sin \omega t}$$

(c) Magnetic field due to displacement current

$= \underline{\underline{0}}$ by symmetry

since $\vec{J}_d$ is <u>radial</u> $\propto \hat{r}$

4) <u>Wire gap</u>

(a) Wire "fat" $\Rightarrow$ surface large

$$\Rightarrow \quad \vec{E} = \frac{\sigma}{\varepsilon_0}\hat{z} \; ; \; \sigma = \frac{Q}{\pi a^2} \; ; \; Q = It$$

(field's constant — it's constant)

$$\text{so} \quad \boxed{\vec{E} = \frac{It}{\pi \varepsilon_0 a^2}\hat{z}}$$

Amperian loop radius s in gap:

$$\oint \vec{B}\cdot d\vec{\ell} = \oint (\vec{J}_f + \vec{J}_d)\cdot d\vec{a}$$

($J_f \to 0$)

$$B\, 2\pi s = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \pi s^2$$

$$\Rightarrow \quad \boxed{\vec{B}(s,t) = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}}$$

(b) $u_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$

$$= \frac{1}{2}\left[\varepsilon_0\left(\frac{It}{\pi\varepsilon_0 a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right]$$

$$= \frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + \frac{s^2}{4}\right]$$

used $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = \frac{1}{\mu_0}\left(\frac{It}{\pi\varepsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)\left(-\hat{s}\right)$$

$$\boxed{\vec{S} = \frac{-I^2 t s}{2\pi^2\varepsilon_0 a^4}\,\hat{s}}$$

used $\boxed{\hat{z}\times\hat{\phi} = -\hat{s}}$

(c) Conservation of energy $\frac{\partial u_{em}}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$

$$\frac{\partial u_{em}}{\partial t} = \frac{I^2 t}{\pi^2\varepsilon_0 a^4} \qquad -\vec{\nabla}\cdot\vec{S} = \frac{I^2 t}{2\pi^2\varepsilon_0 a^4}\,\nabla\cdot(s\hat{s})$$

note $\nabla\cdot(s\hat{s}) = \frac{1}{s}\frac{\partial}{\partial s}(s^2) = 2$

$$\Rightarrow -\vec{\nabla}\cdot\vec{S} = \frac{I^2 t}{\pi^2\varepsilon_0 a^4} \quad \checkmark$$