PHY4324–Electromagnetism II

Fall 2011

Test 2 – 55 minutes

Nov. 7, 2011

No other materials except calculators allowed. If you can't do one part of a problem, solve subsequent parts in terms of unknown answer–define clearly. **Do 2 of 4 problems, CLEARLY indicating which you want graded by circling the problem number!**. *Each problem is worth 15 pts., for maximum of 30 points.*

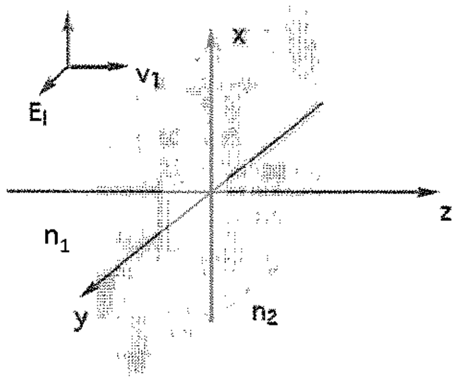


Figure 1: Plane wave in medium 1 traveling along z incident from left on interface to medium 2.

1. **Plane wave.** Suppose the xy-plane forms the boundary between two linear media with indices n_1 (take $\mu = \mu_0$ for both media), n_2. A plane wave $\mathbf{E}_I(z,t) = \mathbf{E_0} e^{ik_1 z - \omega t}\hat{y}$ traveling in the z direction and polarized in the $\hat{y}$ direction approaches the interface from the left (figure above).

   (a) (4 pts.) Calculate $\mathbf{B}_I(z,t)$ and indicate its direction in the figure.

   (b) (4 pts.) Write down the form of the reflected electric and magnetic field, $\mathbf{E}_R(z,t)$, $\mathbf{B}_R(z,t)$ and the transmitted electric and magnetic fields, $\mathbf{E}_T(z,t)$, $\mathbf{B}_T(z,t)$ respectively.

   (c) (4 pts.) Write down boundary conditions for the fields at z=0 and use them to calculate the reflected and transmitted $\mathbf{E}$ fields in terms of incident field, indicating the directions of these fields in the figure.

   (d) (3 pts.) If medium 2 is a conductor with conductivity σ and permeability μ_0, permittivity ϵ instead of a dielectric, what is the new form of the tranmitted electric field? (For full credit give the mathematical form, the amplitude of the electric field in region 2 in terms of the incident amplitude, and the dispersion relation.)

2. **Parallel-plate waveguide.** A waveguide consists of two infinite parallel plates, as shown. Consider TM waves propagating in the $+z$ direction, and assume that the fields are *uniform* along the x direction.

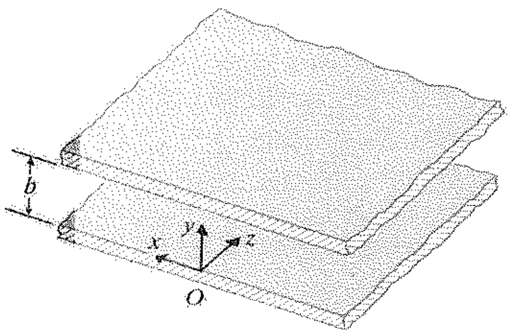


Figure 2: Infinite parallel-plate waveguide

(a) (4 pts.) State the boundary conditions on the electric and magnetic fields at the inner surfaces of the plates.

(b) (4 pts.) Write the equation for the z component of the electric field amplitude $E_z^0(y)$, and state the solution, consistent with the boundary conditions you found in (a). (*Hint:* even if you don't know the equation, you should be able to guess the solution.)

(c) (4 pts.) Using Maxwell's equations, find the transverse components $B_x^0(y)$ and $E_y^0(y)$.

(d) (3 pts.) Find the dispersion of the TM modes $\mathbf{k}(\omega)$ and state the cutoff frequency.

3. **Point charge.** The electric field of a moving point charge q moving at constant velocity $v\hat{x}$ has the form

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\frac{1-(v/c)^2}{(1-(v/c)^2\sin^2\theta)^{3/2}}\hat{R}, \tag{1}$$

where θ is the angle between the vector $\mathbf{r}$ and the x axis, and $\mathbf{R}$ is the vector from the particle to the field point $\mathbf{r}$ at the present time.

(a) (4 pts.) Sketch the electric field lines for the case v close to c.

(b) (4 pts.) Find the magnetic field.

(c) (4 pts.) Use the result (1) to derive the expression for the electric field of an infinite line charge: assume there are infinitely many charge elements $dq = \lambda dx$, each moving with velocity v, and integrate the charge densities to find the total electric field a distance s from the axis. *Hint:* You may need $\int_0^\pi d\theta \sin\theta/(1-b^2\sin^2\theta)$.

(d) (3 pts.) Go back to pt. charge moving with velocity v. At $t=0$ the charge is located at the origin $x=0$. How much energy from the point charge is passing through the plane $\perp$ to the x-axis at $x=a$? Set up in terms of a θ-integral which you need not evaluate. *Hint:* you may need $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$.

4. **Wire loop.**

(a) (4 pts.) Using the identities

i. $\nabla \rho = -\frac{1}{c}\dot{\rho}\hat{\mathscr{r}}$ (here $\rho = \rho(\mathbf{r}', t_r)$)

ii. $\nabla \cdot (\hat{\mathscr{r}}/\mathscr{r}) = 1/\mathscr{r}^2$

iii. $\nabla(1/\mathscr{r}) = -\hat{\mathscr{r}}/\mathscr{r}^2$

iv. $\nabla \cdot (\hat{\mathscr{r}}/\mathscr{r}^2) = 4\pi\delta^3(\vec{\mathscr{r}})$,

and the retarded form of the potential

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}, \tag{2}$$

show explicitly that $\Box^2 V = -\frac{1}{\epsilon_0}\rho$.

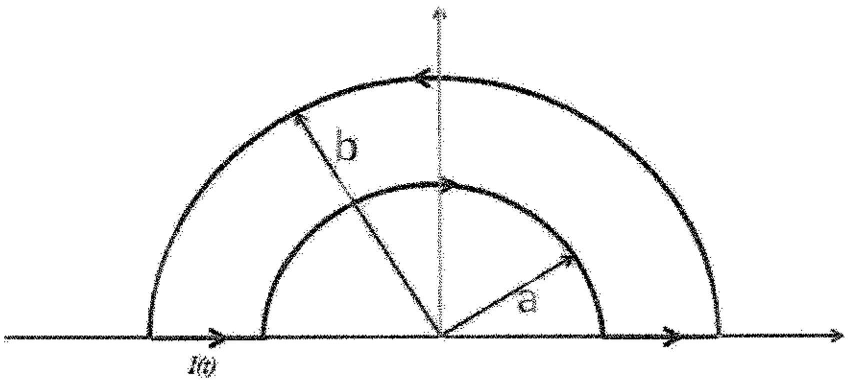


Figure 3: Wire loop with current $I(t)$ as shown.

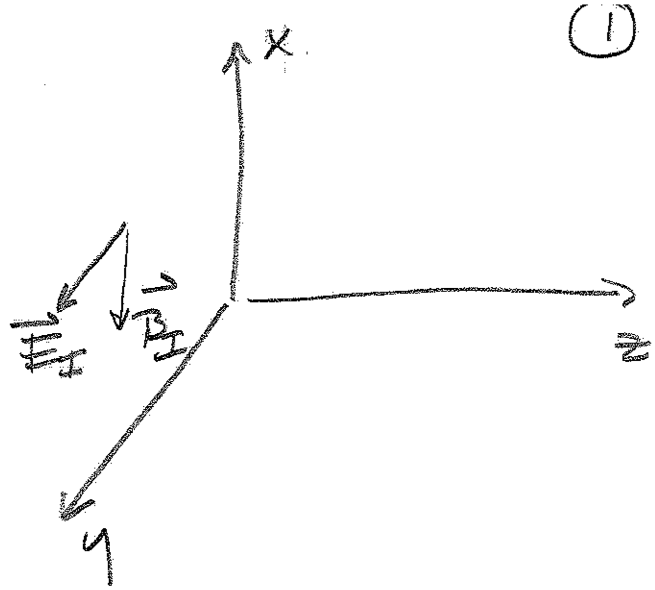
The next questions are based on the figure, where a wire is bent into a loop as shown in the figure. Label the two arcs and two segments clearly with numbers 1-4. The goal is to calculate the potentials and fields at $\vec{r} = 0$ (center of arcs) for a current increasing as $I = kt$, where k is a constant. There is no net charge on the wire.

(b) (4 pts.) Calculate $\mathbf{A}(\mathbf{r} = 0, t)$ and $V(\mathbf{r} = 0, t)$.

(c) (4 pts.) Calculate $\mathbf{E}(\mathbf{r} = 0, t)$.

(d) (3 pts.) Explain how you would go about calculating $\mathbf{B}(\mathbf{r} = 0, t)$, but do not do it!

1) Plane wave



a) For plane wave in SI units, $|\vec{B}| = \frac{|\vec{E}|}{v}$

where v is velocity of wave in medium.

$\vec{E} \times \vec{B}$ must be in the direction of propagation (Poynting vector), so $\vec{B}_I \propto -\hat{x}$

$$\Rightarrow \quad \vec{B}_I(z,t) = \frac{E_{0I}}{v_1} e^{i(k_1 z - \omega t)} (-\hat{x})$$

b)

$$\vec{E}_R(z,t) = E_{0R} \, e^{i(-k_1 z - \omega t)} \hat{y}$$

$$\vec{B}_R(z,t) = \frac{E_{0R}}{v_1} e^{i(-k_1 z - \omega t)} \hat{x}$$

note $\vec{B}$ changes sign on reflection!

$$\boxed{\begin{aligned} v_1 &= \frac{c}{n_1} \\ v_2 &= \frac{c}{n_2} \\ k_1 &= \frac{\omega}{v_1} \\ k_2 &= \frac{\omega}{v_2} \end{aligned}}$$

$$\vec{E}_T(z,t) = E_{0T} \, e^{i(k_2 z - \omega t)} \hat{y}$$

$$\vec{B}_T(z,t) = \left(\frac{E_{0T}}{v_2}\right) e^{i(k_2 z - \omega t)} (-\hat{x})$$

where

$$E_{0R} = \frac{1-\beta}{1+\beta} E_{0I} \; ; \quad E_{0T} = \frac{2}{1+\beta} E_{0I}$$

$\beta = n_2/n_1$ since $\mu_1 = \mu_2 = \mu_0$ given

c) parallel fields only (normal incidence)

$$\vec{E}_1^{\,\parallel} = \vec{E}_2^{\,\parallel} \; ; \quad \frac{1}{\mu_1} \vec{B}_1^{\,\parallel} = \frac{1}{\mu_2} \vec{B}_2^{\,\parallel}$$

$$\Rightarrow \quad \vec{B}_1^{\,\parallel} = \vec{B}_2^{\,\parallel}$$

Substituting forms from b) gives

$$\frac{E_{0R}}{E_{0I}} \quad , \quad \frac{E_{0T}}{E_{0I}} \quad \text{as in b)}$$

magnitudes of B-ratios same except

$$\frac{B_{0T}}{B_{0I}} = \frac{E_{0T}}{E_{0I}} \cdot \frac{v_1}{v_2}$$

d) $$\vec{E}_T = E_{0T}\, e^{i(\tilde{k}_2 z - \omega t)}\ \hat{y}$$

$$\tilde{k}_2^{\,2} = \mu_0 \varepsilon \omega^2 + i \mu_0 \sigma \omega$$

$$E_{0T} = \frac{2}{1+\tilde{\beta}} E_{0I} \qquad \tilde{\beta} = \frac{v_1 \tilde{k}_2}{\omega}$$

complex

2) Parallel-plate "waveguide" TM modes

$$E_z = E_z^0 \exp i(kz - \omega t)$$

a) B.C. are

$E_z^0(y) = 0$ at $y = 0, b$

$B_z = 0$ everywhere (TM)

b) $$\left(\frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right) E_z^0 = 0$$

soln $$E_z^0 = E_0 \sin \frac{n\pi y}{b}$$

$$\Rightarrow \boxed{-\left(\frac{n\pi}{b}\right)^2 + \left(\frac{\omega}{c}\right)^2 = k^2}$$

c) Maxwell $\Rightarrow$

$$B_x^0 = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k \frac{\partial B_z}{\partial x} - \frac{\omega}{c}\frac{\partial E_z}{\partial y}\right)$$

$$= \frac{-iE_0}{\left(\frac{\omega}{c}\right)^2 - k^2} \frac{\omega}{c}\left(\frac{n\pi}{b}\right) \cos \frac{n\pi y}{b}$$

$$E_y^0 = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x}\right)$$

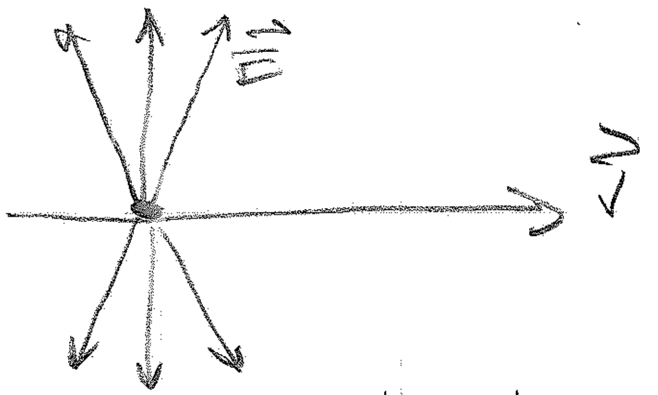
$$= \frac{ikE_0}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(\frac{n\pi}{b}\right) \cos \frac{n\pi y}{b}$$

d) From b) $$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$k = 0$ gives cutoff $\boxed{\omega_1 = \frac{c\pi}{b}}$ $n = 1$ lowest

3) $$\vec{E} = \frac{1}{4\pi\varepsilon_0 R^2}\frac{1-(v/c)^2}{\left(1-\left(\frac{v}{c}\right)^2\sin^2\theta\right)^{3/2}}\hat{R}$$

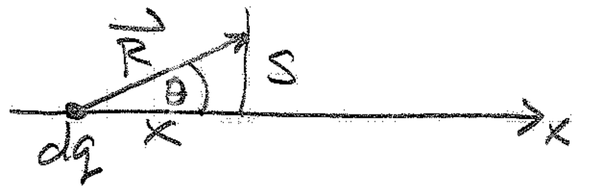
a)



Density of field lines higher ⊥ to direction of motion

b) $$\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$$

c) $$dq = \lambda\, dx$$



$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left(1-v^2/c^2\right)\int_{-\infty}^{\infty}\frac{d\hat{R}}{R^2}\frac{dx}{\left(1-\left(\frac{v}{c}\right)^2\sin^2\theta\right)^{3/2}}$$

use: $$s = R\sin\theta\ ;\quad \frac{1}{R^2} = \frac{\sin^2\theta}{s^2}$$

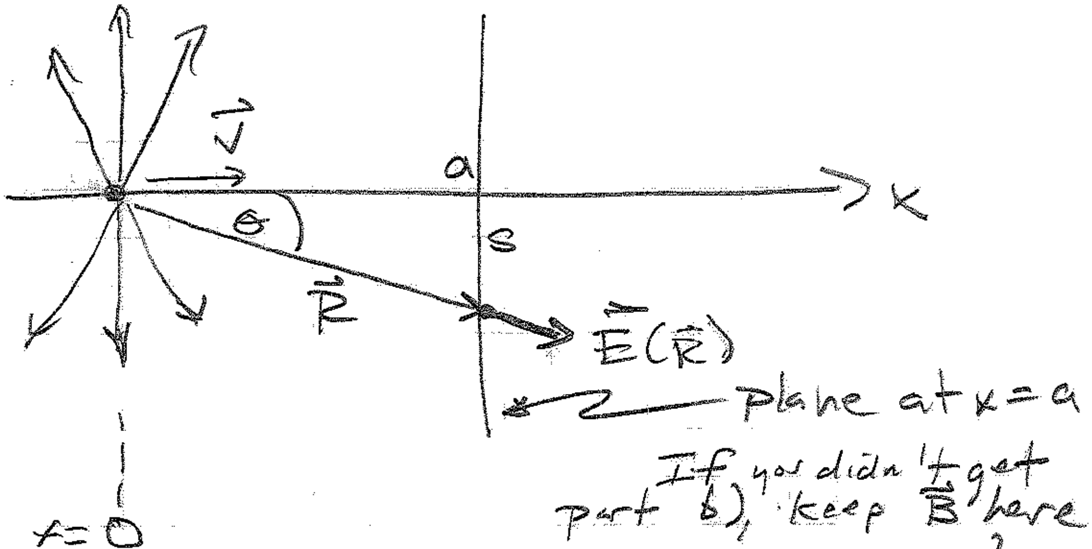
$$\frac{x}{s} = \cot\theta\ ;\ dx = s\csc^2\theta\, d\theta = \frac{s}{\sin^2\theta}d\theta$$

$$\frac{1}{R^2}dx = \frac{d\theta}{s}\ ;\quad \left(\hat{R}\right)_s = \sin\theta$$

$$\Rightarrow \vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left(1-\left(\frac{v}{c}\right)^2\right)\left(\frac{\hat{s}}{s}\right)\underbrace{\int_0^{\pi}d\theta\frac{\sin\theta}{\left(1-\left(\frac{v}{c}\right)^2\sin^2\theta\right)^{3/2}}}_{2\cdot\left(1-(v/c)^2\right)^{-1}}$$

$$= \frac{\lambda}{2\pi\varepsilon_0 s}\,\hat{s}$$ as for line chg. at rest.

d)



If you didn't get part b), keep $\vec{B}$ here

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0 c^2}\,\vec{E}\times(\vec{v}\times\vec{E})$$

BAC-CAB! $\Rightarrow$ $= \varepsilon_0\left(\vec{v}E^2 - \vec{E}(\vec{v}\cdot\vec{E})\right)$

Power thru plane $= \int_{plane} \vec{S}\cdot d\vec{a} = \varepsilon_0\int (vE^2 - E_x^2 v)\,2\pi s\,ds$

$$\boxed{d\vec{a} = 2\pi s\,ds\,\hat{x}}$$

$$\boxed{\begin{aligned} E_x &= E\cos\theta \\ \Rightarrow\; E^2 - E_x^2 &= E^2\sin^2\theta \end{aligned}}$$

$$P = 2\pi v\varepsilon_0\int E^2(\theta)\cdot(a\tan\theta)\left(\frac{a\,d\theta}{\cos^2\theta}\right)\sin^2\theta$$

$$\boxed{\begin{aligned} s &= a\tan\theta \\ ds &= \frac{a\,d\theta}{\cos^2\theta} \end{aligned}}$$

$$E = \frac{q}{4\pi\varepsilon_0 R^2}\,\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}$$

Full credit

$$\text{so } P = \frac{2\pi v q^2}{(4\pi)^2\varepsilon_0}\,\frac{(1-\beta^2)^2}{a^2}\underbrace{\int_0^{\pi} d\theta\,\frac{\sin^3\theta\cos\theta}{(1-\beta^2\sin^2\theta)^{3/2}}}_{\frac{1}{4}(1-\beta^2)^{-2}}$$

$$= \frac{q^2 v}{32\pi\varepsilon_0 a^2}$$

4) Wire loop

(a) Show $\Box^2 V = -\frac{1}{\varepsilon_0}\rho$

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Start with

$$\nabla V = \frac{1}{4\pi\varepsilon_0}\int d\tau' \left(\frac{\nabla\rho(\vec{r}\,', t_r)}{\imath} + \rho(\vec{r}\,', t_r)\vec{\nabla}\frac{1}{\imath} \right)$$

$$= \frac{1}{4\pi\varepsilon_0}\int d\tau' \left(\underset{\text{i}}{\frac{-\frac{1}{c}\dot{\rho}\,\hat{\imath}}{\imath}} + \rho\underset{\text{iii}}{\left(\frac{-\hat{\imath}}{\imath^2}\right)} \right)$$

$$\text{So } \nabla\cdot\nabla V = \frac{1}{4\pi\varepsilon_0}\int d\tau' \Big[-\frac{1}{c}\frac{(\nabla\dot{\rho})\hat{\imath}}{\imath} - \frac{1}{c}\dot{\rho}\,\vec{\nabla}\left(\frac{\hat{\imath}}{\imath^2}\right) + \vec{\nabla}\rho\left(\frac{-\hat{\imath}}{\imath^2}\right) - \rho\nabla\left(\frac{\hat{\imath}}{\imath^2}\right) \Big]$$

$$= \frac{1}{4\pi\varepsilon_0}\int d\tau' \Big[-\frac{1}{c^2}\frac{\ddot{\rho}}{\imath} - \cancel{\frac{1}{c}\frac{\dot{\rho}}{\imath^2}} + \cancel{\frac{1}{c}\dot{\rho}\frac{1}{\imath^2}} - \rho\, 4\pi\, \underset{\text{iv}}{\delta^{(3)}(\vec{\imath})} \Big]$$

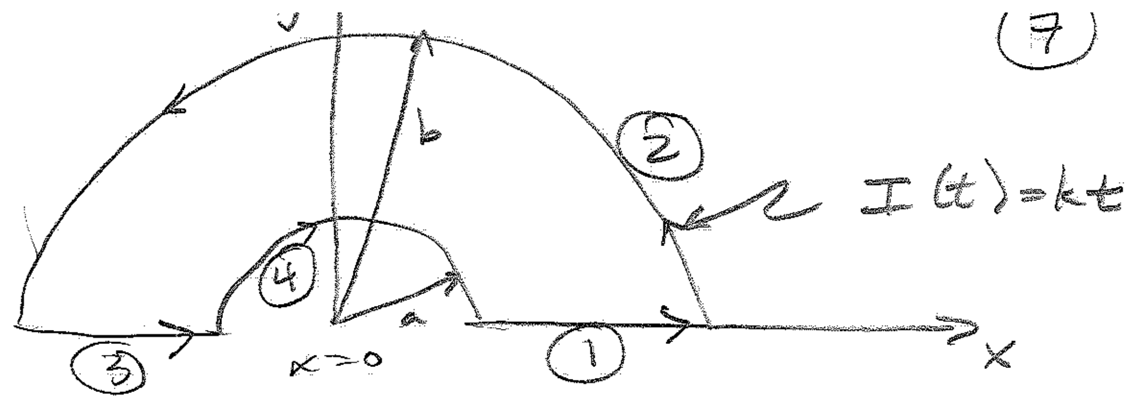
$$= \frac{1}{4\pi\varepsilon_0}\int d\tau' \left(-\frac{1}{c^2}\frac{\ddot{\rho}}{\imath} \right) - \frac{1}{\varepsilon_0}\rho(\vec{r}, t)$$

δ-fctn sets $\vec{r} = \vec{r}\,'$

$$\text{Also: } \frac{\partial^2}{\partial t^2}V = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\ddot{\rho}(\vec{r}\,', t_r)}{\imath}$$

$$\text{since } \frac{\partial}{\partial t}t_r = \frac{\partial}{\partial t}(t - \imath/c) = 1$$

$$\text{So } \Box^2 V = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V = -\frac{1}{\varepsilon_0}\rho(\vec{r}, t) \quad !$$



(b) $$\vec{A}(\vec{r}=0,t) = \frac{\mu_0}{4\pi}\oint \frac{I(t_r)\,dl'}{\mathscr{r}}$$

$$= \frac{\mu_0}{4\pi} k \oint \left(\frac{t - \mathscr{r}/c}{\mathscr{r}}\right) dl'$$

$$= \frac{\mu_0 k}{4\pi} t \oint \frac{d\vec{l}}{\mathscr{r}}$$ since 2nd term $\propto \oint dl' = 0$!

So we need $\mathscr{r}$ on 1 ... 4

Since $\vec{r} = 0$, $\mathscr{r} = r'$

1 $r' = x'$ 2 $r' = b$ 3 $r' = -x'$ 4 $r' = a$

$$\Rightarrow \oint \frac{d\vec{l}}{\mathscr{r}} = \int_a^b \frac{dx'}{x'} + \cancel{\int_0^\pi \frac{b\,d\theta}{b}} + \int_{-b}^{-a}\left(\frac{dx'}{-x'}\right) + \cancel{\int_\pi^0 \frac{a\,d\theta}{a}}$$

cancel

$$= 1 + 3 = 2\int_a^b \frac{dx'}{x'} = 2\ln\frac{b}{a}\,\hat{x}$$

$$\Rightarrow \vec{A}(\vec{r}=0,t) = \frac{\mu_0 k t}{2\pi}\ln\frac{b}{a}\,\hat{x}$$

and $V(\vec{r}=0,t) = 0$ since $\rho = 0$

c) $$\vec{E}(\vec{r}=0,t) = -\nabla V + \frac{\partial \vec{A}}{\partial t}$$

$$= 0 + \frac{\mu_0 k}{2\pi} \ln\frac{b}{a}\,\hat{x}$$

induced by
t-changing
$\vec{B}$-field!

d) Can't calculate $\vec{B}$
from $\vec{\nabla}\times\vec{A}$ unless we know $\vec{A}$
in a neighborhood of $\vec{r}=0$. So
far we have $\vec{A}$ only at one point!

So go back & calculate $\vec{A}$ for some
neighborhood, take curl OR use Jefimenko:

$$\vec{B}(\vec{r}=0,t) = -\frac{\mu_0}{4\pi}\int\left[\frac{I(\vec{r}',t_r)}{r'^2} + \frac{\dot{I}(\vec{r}',t_r)}{c r'}\right] d\vec{l}' \times \hat{r}'$$