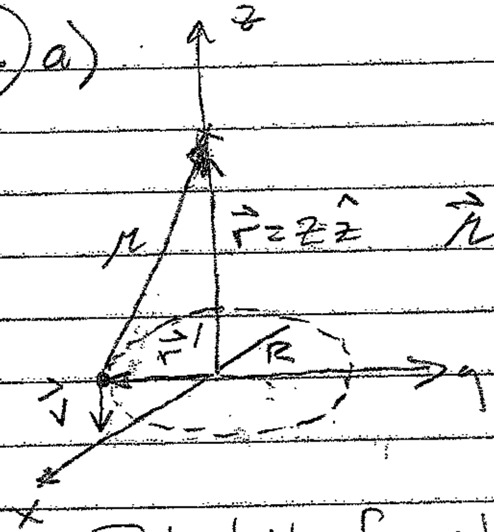
Solutions to test 2

1. a)



$\vec{r}\,' = R(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$

$\vec{V} = \dot{\vec{r}}\,' = \omega R(-\sin\omega t\,\hat{x} + \cos\omega t\,\hat{y})$

$\vec{r} = z\hat{z}$ $\quad \vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = z\hat{z} - R(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$

Note 1) we need $\vec{\mathscr{r}}(t_r)$ retarded time

2) $\vec{\mathscr{r}}(t_r) \cdot \vec{V}(t_r) = 0$

Potentials for pt. chg. are [G 10.39 + 10.40]

1) $V(\vec{r}, t) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{qc}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}} = \dfrac{q}{4\pi\varepsilon_0 \mathscr{r}} = \dfrac{q}{4\pi\varepsilon_0\sqrt{z^2+R^2}}$

$\mathscr{r}$ is ind. of time so it is ind. of retarded time! Similarly

2) $\vec{A}(\vec{r}, t) = \dfrac{\mu_0}{4\pi} \dfrac{qc\vec{v}}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}} = \dfrac{q\mu_0\vec{v}(t_r)}{4\pi\mathscr{r}}$

$= \dfrac{q\omega R}{4\pi\varepsilon_0 c^2\sqrt{z^2+R^2}}(-\sin\omega t_r\,\hat{x} + \cos\omega t_r\,\hat{y})$

b) $\vec{a}(t_r) = \ddot{\vec{r}}\,' = -R\omega^2(\cos\omega t_r\,\hat{x} + \sin\omega t_r\,\hat{y})$

$= -\omega^2\vec{r}\,'(t_r)$

c) At origin $\vec{r} = 0$; $\vec{\mathscr{r}} = -\vec{r}\,'$; $\hat{\mathscr{r}} = -(\cos\omega t_r\,\hat{x} + \sin\omega t_r\,\hat{y})$

$\mathscr{r} = R$; $\quad t_r = t - R/c$

We can't just use $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$ since $\vec{E}$ has a sideways (xy plane) component at any given time, and to calculate xy gradients we need info off the z axis. Therefore its necessary to use the Jefimenko expressions for fields of a pt. charge:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}} \cdot \vec{u})^3} \left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a}) \right]$$

$$\vec{B} = \frac{1}{c} \hat{\mathscr{r}} \times \vec{E}$$

where $\vec{u} = c\hat{\mathscr{r}} - \vec{v}$

So $\vec{u} = -c\underbrace{(\cos\omega t_r \hat{x} + \sin\omega t_r \hat{y})}_{\hat{\mathscr{r}}} - \omega R \underbrace{(-\sin\omega t_r \hat{x} + \cos\omega t_r \hat{y})}_{\hat{z} \times \hat{\mathscr{r}}}$

BAC-CAB! $\vec{\mathscr{r}} \times (\vec{u} \times \vec{a}) = -\underbrace{(\vec{u} \cdot \vec{\mathscr{r}})}_{Rc}\vec{a} + \underbrace{(\vec{a} \cdot \vec{\mathscr{r}})}_{R^2\omega^2}\vec{u}$

$$(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a}) = c^2\vec{u} - Rc\vec{a}$$

$$\Rightarrow \boxed{\begin{aligned} \vec{E}(\vec{r}=0,t) &= \frac{q}{4\pi\varepsilon_0 R^2 c^2} \left[(R^2\omega^2 - c^2)\hat{\mathscr{r}} - R\omega c(\hat{z} \times \hat{\mathscr{r}}) \right] \\ \vec{B}(\vec{r}=0,t) &= \frac{q\omega}{4\pi\varepsilon_0 R c^2} \hat{z} \end{aligned}}$$

1 d ring of charge $\lambda = \lambda_0 \sin\theta/2$

$$V(\vec{R}=0, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\theta, t_r)}{R} R\,d\theta'$$

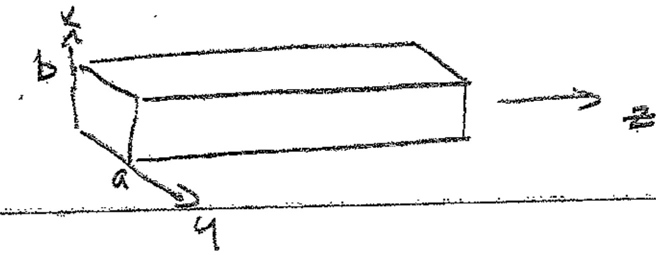
$$= \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{\lambda_0 \sin\left[(\theta' - \omega t_r)/2\right] R\,d\theta'}{R}$$

$t_r = t - \frac{R}{c}$

$$= \frac{\lambda_0}{4\pi\varepsilon_0}\int_0^{2\pi} d\theta' \left(\sin\theta'/2 \cos\frac{\omega t_r}{2} - \sin\frac{\omega t_r}{2}\cos\frac{\theta'}{2}\right)$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0}\cos\frac{\omega t_r}{2}\underbrace{\int_0^{2\pi}\sin\theta'/2\,d\theta'}_{4}$$

$$= \frac{\lambda_0 \cos\omega(t - R/c)/2}{2\pi\varepsilon_0}$$



2) Waveguide

a) $\vec{E}^{\parallel}$ must be continuous at boundary. Since $E=0$ in conductor, $\vec{E}^{\parallel}=0$. For geometry shown, this means $E_z=0$ at $x=0,b$ and $y=0,a$.

b) $E_z(x,y) = X(x)Y(y)$

Substitute into $\left(\partial_x^2 + \partial_y^2 + \frac{\omega^2}{c^2} - k^2\right)E_z = 0$

$$\Rightarrow Y\partial_x^2 X + X\partial_y^2 Y + \left(\frac{\omega^2}{c^2} - k^2\right)XY = 0$$

$$\Rightarrow \frac{1}{X}\partial_x^2 X + \frac{1}{Y}\partial_y^2 Y + \left(\frac{\omega^2}{c^2} - k^2\right) = 0$$

$$\Rightarrow \frac{1}{X}\partial_x^2 X = -k_x^2$$

$$\frac{1}{Y}\partial_y^2 Y = -k_y^2$$

$$\boxed{\frac{\omega^2}{c^2} - k^2 - k_x^2 - k_y^2 = 0}$$

soln: $X = A\sin k_x x + B\cos k_x x$

B.C. $X(0) = X(b) = 0 \Rightarrow B = 0$

$k_x = m\pi/b \quad m = 1, 2, \ldots$

Note $m=0$ is <u>not</u> allowed; gives $X=0$ identically

So $X(x) = A \sin m\pi x/b \quad m = 1, 2 ..$

same for $Y(y) = A' \sin n\pi y/a \quad n = 1, 2 ..$

$$E_z(x,y) = E_{z0} \sin \frac{m\pi x}{b} \sin \frac{n\pi y}{a}$$

$$m, n = 1, 2 \ldots$$

Dispersion is now $k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{b}\right)^2 - \left(\frac{n\pi}{a}\right)^2}$

$$= \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_{mn}}{c}\right)^2}$$

$$\omega_{mn} = c\sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2}$$

c) Minimum propagating frequency (s.t. k is real) is

$$\omega_{11} = \sqrt{\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{a}\right)^2}$$

<u>Larger</u> than TE mode w/ same geometry since n <u>or</u> $m = 0$ is allowed in that case.

d) Phase vel. $v_p = \frac{\omega}{k} = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_{mn}}{c}\right)^2}}$

$$= \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}}$$

Group velocity $\frac{\partial \omega}{\partial k} = \left(\frac{\partial k}{\partial \omega}\right)^{-1}$

$$= c\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2} < c \,!$$

3) Drop tildes "~" and take Re parts at end:

Note $\vec{E}_{1R}$ is full reflected wave in I on I

a) I : $\vec{E}_I = \vec{E}_{1I}\, e^{i(kz-\omega t)} + \vec{E}_{1R}\, e^{i(-kz-\omega t)}$ ($k=\omega/c$ (air))

II $\vec{E}_{II} = \vec{E}_{2T}\, e^{i(k_2 z-\omega t)} + \vec{E}_{2R}\, e^{-i(k_2 z-\omega t)}$

III $\vec{E}_{III} = \vec{E}_{3T}\, e^{i(kz-\omega t)}$

Note $k_1 = k_3 = k = \omega/c$ since we're told $n_{1,3}=1$,
$k_2 = k/n$

b) Gen'l B.C. are $D^\perp, B^\perp$ continuous
$E^{\parallel}, H^{\parallel}$ continuous

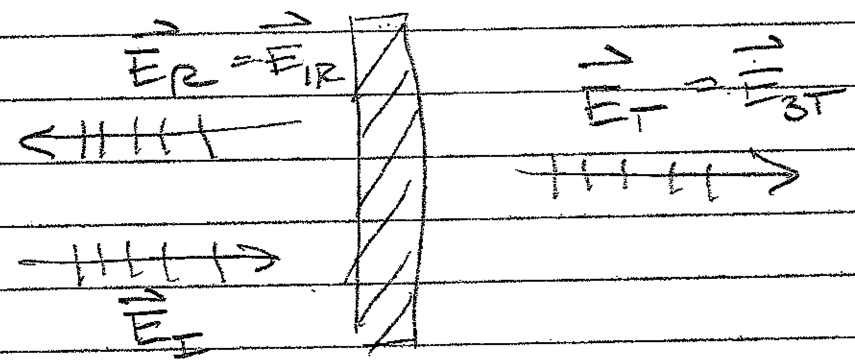
Here due to normal incidence we have no $\perp$ components; also $H^{\parallel} = B^{\parallel}$ since $\mu = \mu_0$.

c) at I-II interface $z=0$

c1) $\vec{E}_{1I} + \vec{E}_{1R} = \vec{E}_{2T} + \vec{E}_{2R}$

at $z=d$ c2) $e^{ik_2 d}\vec{E}_{2T} + e^{-ik_2 d}\vec{E}_{2R} = e^{ikd}\vec{E}_{3T}$

Note there are 2 interfaces but the total reflection coeff. R is the reflected wave in region I (which includes the reflected wave from the interface at d), similarly the transmission makes the final transmission amplitude



So we would solve for

$$R = \left|\frac{E_{1R}}{E_{I}}\right|^2 \qquad T = \left|\frac{E_{3T}}{E_{1I}}\right|^2$$

To solve we have 4 eqns with 5 unknowns: $E_{1I}, E_{1R}, E_{2T}, E_{2R}, E_{3T}$

(eqns are c1) and c2) with magnetic analogs with $B_\alpha = E_\alpha / v_\alpha$)

So we can solve for all 4 reflected + transmitted amplitudes in terms of E_{1I}

Prob. 4. Gauge transform

a) $$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$$

$$\vec{A} = -\frac{bt}{2\pi\varepsilon_0 s}\hat{s} \qquad V = 0$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = \frac{b}{2\pi\varepsilon_0 s}\hat{s}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = 0$$

Field of infinite line charge $b = \lambda$, chg./unit length.

b) $$\vec{A}' = -\frac{1}{2\pi\varepsilon_0}\frac{bt}{s}\hat{s} + \nabla\Lambda = 0$$

$$\nabla \ln s = \frac{\hat{s}}{s}$$

$$\Lambda = \frac{bt \ln s}{2\pi\varepsilon_0}$$

$$V' = 0 - \frac{\partial\Lambda}{\partial t} = -\frac{b}{2\pi\varepsilon_0}\ln s$$

Coulomb gauge $\nabla\cdot\vec{A}' = 0$ since $\vec{A}' = 0$ ✓

Lorentz gauge $\nabla\cdot A' + \mu_0\varepsilon_0\frac{\partial V}{\partial t} = 0$ ✓

c) Can always find a gauge where $V' = 0$:

$$V' = V - \frac{\partial \Lambda}{\partial t} = 0 \implies$$

choose $\Lambda(\vec{r}, t) = \int^t dt' \, V(\vec{r}, t')$

<u>But</u> can't always find a gauge for every physical situation where $\vec{A}' = 0$

$$\vec{A}' = \vec{A} + \nabla \Lambda = 0$$

would $\implies$ $\vec{A} = -\nabla \Lambda$

$$\implies \vec{B} = \nabla \times \vec{A} = -\nabla \times \nabla \Lambda = 0$$

So if (gauge-invariant) magnetic field $\vec{B}$ isn't zero, there is no gauge with $\vec{A}' = 0$.