PHY4605–Introduction to Quantum Mechanics II

Spring 2005

Problem Set 3

Jan. 24, 2005

Due: 31 January, 2005
Reading: PH notes

Remarks: On problem set 1 too many people had continued difficulty with matrix algebra. To solidify these crucial concepts, here's more practice.

1. **Matrix algebra drill**. Given the matrices

$$\underline{A} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \quad ; \quad \underline{B} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 2 & 0 \end{pmatrix} \quad ; \quad \underline{C} = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix} \tag{1}$$

   (a) Find the eigenvalues λ_i and *normalized* eigenvectors $\mathbf{v}_i$, $i = 1, 2, 3$ for $\underline{A}$, $\underline{B}$, and $\underline{C}$. State the degeneracy of each eigenvalue.

   (b) Find the determinants of $\underline{A}$, $\underline{B}$, and $\underline{C}$.

   (c) Recall that, under a change of basis matrices, transform as $\underline{M} \rightarrow \underline{M}' = \hat{U}^{-1}\underline{M}\hat{U}$ and vectors as $\mathbf{v} \rightarrow \mathbf{v}' = \hat{U}^{-1}\mathbf{v}$. Find $\hat{U}$ such that $\underline{A}$ is brought into diagonal form by a change of basis. How is the matrix of transformation $\hat{U}$ related to the eigenvectors of $\underline{A}$? Find the transformed eigenvectors $\hat{U}^{-1}\mathbf{v}_i$. Now answer the same questions for $\underline{C}$.

   (d) Find the inverses of $\underline{A}$ and $\underline{B}$, and state why $\underline{C}$ is not invertible.

   (e) Show that the inverse of the 2D matrix represented by $\underline{M} = a_0\mathbf{1} + \vec{a}\cdot\vec{\sigma}$ is $M^{-1} = D_0^{-1}(a_0\mathbf{1} - \vec{a}\cdot\vec{\sigma})$, with $D_0 = a_0^2 - \vec{a}\cdot\vec{a}$. Here $\mathbf{1}$ is the identity matrix in 2D and σ_x, σ_y, and σ_z are the Pauli matrices. [Hint: properties of the Pauli matrices you may find useful: 1) $\sigma_i^2 = \mathbf{1}$ for any i; 2) $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k$ for $i \neq j$.]

2. **Phase shift in a vector potential.**

   (a) Show that the vector potential outside an infinite solenoid containing a flux Φ has the magnitude $\Phi/2\pi\rho$, where ρ is the perpendicular distance from the axis of the solenoid.

   (b) Show that if $\psi_{\mathbf{A}}$ satisfies

$$i\hbar\frac{\partial\psi_{\mathbf{A}}}{\partial t} = \frac{1}{2m}[\mathbf{p} - e\mathbf{A}(\mathbf{r})]^2\psi_{\mathbf{A}},$$

   then

$$\psi_{\mathbf{A}}(\mathbf{r}, t) = \exp\left(\frac{ie}{\hbar}\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{A}\cdot d\mathbf{s}\right)\psi_{\mathbf{A}=0}(\mathbf{r}, t)$$

   Apparently $\psi_{\mathbf{A}}$ depends on an arbitrary initial point $\mathbf{r}_0$. Comment on this ambiguity.