# Euler-Lagrange Equations for charged particle in a field

The Lagrangian is

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + q(\mathbf{A}\cdot\dot{\mathbf{r}} - \phi)$$

Euler Lagrange Equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}} = \frac{\partial L}{\partial \mathbf{r}},$$

so calculate left and right hand sides separately:

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{r}} &= q\frac{\partial}{\partial \mathbf{r}}(\mathbf{A}\cdot\dot{\mathbf{r}}) - q\frac{\partial \phi}{\partial \mathbf{r}} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}} &= m\ddot{\mathbf{r}} + q\frac{d}{dt}\mathbf{A}\end{aligned}$$

**Now** recall $\mathbf{A}$ is the vector potential evaluated at the position of the particle $\mathbf{r}$ at time $t$. The particle is following a trajectory $\mathbf{r}(t)$, so $\mathbf{A} = \mathbf{A}(\mathbf{r}(t), t)$. The total time derivative thus gives two terms,

$$\frac{d}{dt}\mathbf{A} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{A},$$

or, to be completely clear, for a given component $A_i$,

$$\frac{d}{dt}A_i = \frac{\partial A_i}{\partial t} + \frac{\partial r_j}{\partial t}\cdot\frac{\partial}{\partial r_j}A_i,$$

where a sum over repeated indices is implied. So, putting the Euler-Lagrange equation together for index $i$ gives

$$\begin{aligned}m\ddot{r}_i &= -q\frac{\partial}{\partial r_i}\phi - q\frac{\partial A_i}{\partial t} + q\left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)\dot{r}_j \\ &= q\left(\mathbf{E} + \dot{\mathbf{r}}\times\mathbf{B}\right)_i,\end{aligned}$$

which is the $ith$ component of the Lorentz force. To convince yourself that the last term in parentheses really turns into the $\dot{\mathbf{r}}\times\mathbf{B}$ term, evaluate

$$\begin{aligned}(\dot{\mathbf{r}}\times\mathbf{B}) &= (\dot{\mathbf{r}}\times(\nabla\times\mathbf{A}))_i = \epsilon_{ijk}\,\dot{r}_j\,(\nabla\times\mathbf{A})_k = \epsilon_{ijk}\,\dot{r}_j\,\epsilon_{k\ell m}\frac{\partial}{\partial r_\ell}A_m \\ &= (\delta_{i\ell}\delta_{km} - \delta_{im}\delta_{k\ell})\dot{r}_j\frac{\partial}{\partial r_\ell}A_m \\ &= \dot{r}_j\frac{\partial}{\partial r_i}A_j - \dot{r}_j\frac{\partial}{\partial r_j}A_i = \left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)\dot{r}_j \quad QED\end{aligned}$$

(I used the cyclic property of the Levi-Civita symbol $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$, and the identity $\epsilon_{ijk}\epsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}$.)