2 Charged Particle in a Magnetic Field

2.1 Classical Particle in B-field

Want to remind ourselves how to write classical Hamiltonian describing charged particle moving in **E** and **B** fields. I will sketch the more thorough review given by, e.g. Gasirowicz in Ch. 16 or Peebles in Sec. 2-19. Griffiths unfortunately does not treat this subject.

Lagrangian for charge q moving in an electromagnetic field is (SI units)

$$L = \frac{m\dot{\mathbf{r}}^2}{2} + q(\mathbf{A}\cdot\dot{\mathbf{r}} - \phi), \tag{1}$$

where $\dot{\mathbf{r}}$ is velocity, $\mathbf{A}$ is vector potential and ϕ scalar potential. Shouldn't swallow this, but check that Euler-Lagrange eqns. $\partial L/\partial r_\alpha = (d/dt)\partial L/\partial \dot{r}_\alpha$ produce classical Lorentz force eqns. we know & love from intro physics. We can do this w/ straightforward but tedious algebra if we remember the classical relations between the fields and the potentials:

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{2}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}. \tag{3}$$

Eqns. of motion become (Lorentz force!)

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{4}$$

The fields **E** and **B** are invariant under *gauge tranformations* of the potentials **A** and ϕ:

$$\phi \to \phi' = \phi - \frac{\partial\chi}{\partial t} \tag{5}$$

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\chi \tag{6}$$

so the eqns. of motion (4)are manifestly *gauge invariant* because they contain only fields, even though the Lagrangian L apparently does not.

Hamiltonian of system now defined as

$$H[\mathbf{p},\mathbf{r}] = \mathbf{p}\cdot\dot{\mathbf{r}} - L, \tag{7}$$

with *canonical momentum*

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + q\mathbf{A}. \tag{8}$$

Replacing velocities with momenta yields

$$\star\star\star \quad \boxed{H = \tfrac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + q\phi} \quad \star\star\star \tag{9}$$

Note: In QM have not yet worked out corresponding formalism for Lagrangian– we'll use different principle to derive Hamiltonian, which will turn out to be exactly (9). Note we will be interpreting the *canonical* momentum $\mathbf{p}$ as $-i\hbar\nabla$, following the prescription to replace the canonical momentum in the classical theory by $-i\hbar\nabla$ to preserve the *canonical commutation relations* $[x,p] = i\hbar$. To convince yourself this is really ok, see Griffiths, prob. 4.59. See Peebles Ch. 2 for a more complete explanation.

2.2 Gauge Invariance in QM

Reminder: unitary transformations in QM

unitary operator $\hat{U} \implies \hat{U}^\dagger = \hat{U}^{-1}$.

If system described by states $|\psi\rangle$, operators $\hat{Q}$, *physically equivalent* representation given by

$$|\psi'\rangle = \hat{U}|\psi\rangle, \ldots \quad ; \qquad \hat{Q}' = \hat{U}\hat{Q}\hat{U}^{-1}. \tag{10}$$

Why? Because matrix elements of all operators identical:

$$\begin{aligned}\langle\psi'|\hat{Q}'|\phi'\rangle &= \langle\psi|\hat{U}^{-1}(\hat{U}\hat{Q}\hat{U}^{-1})\hat{U}|\phi\rangle \\ &= \langle\psi|\hat{Q}|\phi\rangle\end{aligned} \tag{11}$$

QM gauge transformations

Require that a QM gauge tranformation should take wave fctn ψ and potentials $\mathbf{A}$, ϕ to physically equivalent set ψ', $\mathbf{A}'$, ϕ'. Can accomplish this by taking transformation $\psi \to \psi'$ to be unitary, see above. Assume form

$$|\psi'\rangle = \hat{U}|\psi\rangle = e^{iq\chi(\mathbf{r},t)/\hbar}|\psi\rangle, \tag{12}$$

obviously unitary. Note *gauge function* χ is space, time dependent $\Longrightarrow$ called *local* gauge transformation. Now we want Hamiltonian itself, which is observable, to be invariant wrt such transformations *in the sense of Eq.* **??**. Since potential energy is $q\phi$, might guess $H = \mathbf{p}^2/2m + e\phi$, but this is obviously wrong–not gauge invariant, can't get correct eqns. of motion.

Consider gauge-covariant momentum $\Pi \equiv \hat{\mathbf{p}} - q\mathbf{A}$. Note

$$\begin{aligned} \Pi'\psi' \equiv [\hat{\mathbf{p}} - q\mathbf{A} - q\nabla\chi]\psi' &= e^{iq\chi/\hbar}(\hat{\mathbf{p}} - q\mathbf{A})\psi \\ \text{or} \quad \Pi' = e^{iq\chi/\hbar}\Pi e^{-iq\chi/\hbar} & \end{aligned} \tag{13}$$

so Π transforms as we would like under gauge transform, as

$$\Pi' = \hat{\mathbf{p}} - q\mathbf{A}'. \tag{14}$$

Write $H = \hat{T} + q\phi$, require

$$\hat{T}' = e^{iq\chi/\hbar}\hat{T}e^{-iq\chi/\hbar} \tag{15}$$

plus we want $\hat{T}$ to reduce to $\hat{p}^2/2m$ when $\mathbf{A} = 0$ so choose $\hat{T} = \Pi^2/2m$, or

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi \tag{16}$$

as in classical case. With this choice Schrödinger eqn.

$$H'\psi' = i\hbar\frac{\partial\psi'}{\partial t} \tag{17}$$

will have invariant solution ψ' physically equivalent to ψ, soln. of $H\psi = i\hbar(\partial\psi/\partial t)$. Check!

2.3 Bohm-Aharonov Effect

Suppose we have electron inside hollow conductor, or "Faraday cage", with battery which raises potential of cage and region inside, beginning at $t = 0$ and ending at time t. Hamiltonian for electron is $H = H_0 + V(t)$, with $V(t) = -e\phi(t)$. Easy to see only result of varying potential will be varying phase of electronic wave function:

$$\psi(x,t) = \psi_0(x,t)e^{-iS/\hbar}\ , \quad S = \int_0^t V(t')dt' \tag{18}$$

where ψ_0 is wave fctn. in absence of battery. Check:

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= i\hbar\left[\frac{\partial\psi_0}{\partial t}e^{-iS/\hbar} + \psi_0\left(\frac{-i}{\hbar}\right)V(t)e^{-iS/\hbar}\right] \\ &= \left(i\hbar\frac{\partial\psi_0}{\partial t} + V(t)\psi_0\right)e^{-iS/\hbar} \\ &= (H_0 + V(t))\psi_0 e^{-iS/\hbar} = H\psi \end{aligned} \tag{19}$$

where use was made of $H_0\psi_0 = i\hbar\frac{\partial}{\partial t}\psi_0$. But phase shift in single electron's wave fctn. can't affect observables, since physical quantities bilinear in ψ and ψ^*, norm of $e^{-iS/\hbar}$ is 1. This result might encourage sense of complacency, since it's exactly what we expect from classical E & M: since ϕ is constant inside cage, $\mathbf{E} = 0$, so no physical changes.

But in 1959 Aharanov and Bohm[1] looked at variation of above expt. with *two* Faraday cages & two batteries, in which potentials on two cages were different from one another. Naively, might expect that electrons in two arms would experience two different phase shifts, $\Delta\varphi_1 \equiv (-e/\hbar)\int_0^t dt'\phi_1(t')$, and $\Delta\varphi_2 \equiv (-e/\hbar)\int_0^t dt'\phi_2(t')$. Wave fctn. would then be $\psi = \psi_0^1 e^{i\Delta\varphi_1} + \psi_0^2 e^{i\Delta\varphi_2}$. This phase difference, $\Delta\varphi \equiv \Delta\varphi_1 - \Delta\varphi_2$ would then be *observable* because it would shift interference pattern at screen as shown. (To see how $\Delta\phi$ comes in, calculate $|\psi|^2$.) ⋆⋆ *Observable consequence of nonzero ϕ would then be found even though region of space electrons travelled through has electric field $\mathbf{E} = 0$,*

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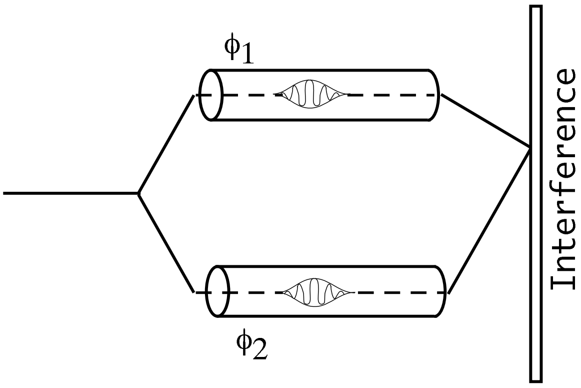
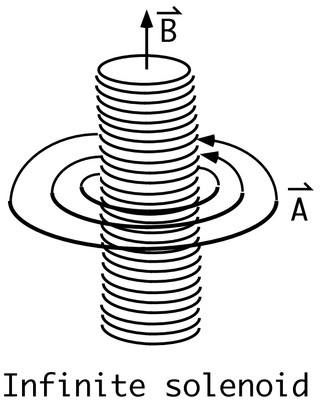
[1] Phys. Rev. 115, 485 (1959)

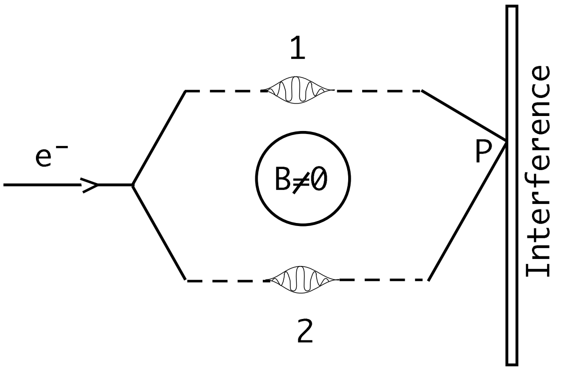


i.e. electrons never subject to classical force! ⋆⋆ Worth noting that obvious interpretation–namely that electromagnetic potentials have independent significance in QM, more "fundamental" than fields perhaps–at least consistent with observation that effect observed (interference pattern for electrons) has no classical analogue.

Problem: cylinders in figure can't be true Faraday cages if electrons pass through them–must be holes, fields could leak in at edges. . . Another, perhaps more convincing, demonstration uses magnetic effects.



Infinite solenoid



Aharonov-Bohm expt.

Pass 2 e^- beams around long solenoid as shown, such that paths fully enclose solenoid, interfere at screen. Note $\mathbf{B} \neq 0$ only inside solenoid, but $\mathbf{A} \neq 0$ everywhere.[2] Might expect (see below) that effect is similar to

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[2] In useful gauge, a solenoidal vector potential can be written (cylindrical coordinates ρ, z, θ)

$$\mathbf{A} = \begin{cases} A_z = A_\rho = 0, \quad A_\theta = B_0\rho/2 & \rho < a \\ A_z = A_\rho = 0, \quad A_\theta = B_0 a^2/(2\rho) & \rho > a \end{cases} \tag{20}$$

electrostatic potential case, i.e. that $\psi = \psi^1 + \psi^2$, and each ψ^α acquires a phase factor $\psi^\alpha = \psi_0^\alpha e^{iS_\alpha/\hbar}$ in the presence of the vector potential. [3] Then a B-dependent shift in interference at screen may produce a measurable effect of magnetic flux, even though e^-'s never pass through it! We may say vector potential $\mathbf{A}$ is more "fundamental" than field $\mathbf{B}$ in QM, or note that this is simply another example of *nonlocality* in QM!

Substitute $\psi_0 e^{iS/\hbar}$ into t-dependent S.-eqn.:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla + e\mathbf{A}\right)^2\psi \tag{22}$$

to find (assuming ψ_0 satisfies zero-field S.-eqn $i\hbar(\partial\psi_0/\partial t) = -\frac{\hbar^2}{2m}\nabla^2\psi_0$) that ψ satisfies the full S-eqn. if we take[4]

$$S(\mathbf{r}) = -e\int^{\mathbf{r}}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}' \tag{23}$$

Easy to see qualitatively now that since $\mathbf{A}$ points in opposite directions on opposite sides of solenoid, beams arrive with different phases $S_1/\hbar$ and $S_2/\hbar$ for nonzero field. Note that the phase difference at a point P on the screen will depend on the difference of the phases accumulated along each trajectory:

$$\Delta\varphi = S_1(P)/\hbar - S_2/\hbar = \frac{-e}{\hbar}\oint_{\text{whole path}}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}' \tag{24}$$

where "whole path" means along path 1 to P and back along path 2 to electron gun. Stokes's theorem then gives

$$\frac{-e}{\hbar}\oint_{\text{whole path}}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}' \;=\; \frac{-e}{\hbar}\int_{\text{area encl.}} d\mathbf{a}\cdot\nabla\times\mathbf{A} \tag{25}$$

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such that

$$\mathbf{B} = \nabla\times\mathbf{A} = \hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\theta) = \begin{cases} B_0 & \rho < a \\ 0 & \rho > a \end{cases} \tag{21}$$

[3] **It is important to realize that we are not talking necessarily about 2 electrons interfering with each other, although we could. We say "beams of electrons", but then in principle we should worry about antisymmetrizing the many-electron wave-function, etc. Think of one electron interfering with itself, in the same sense as a 2-slit experiment. Then ψ^1 is the amplitude the electron follows path 1, and ψ^2 is the amplitude it follows path 2.**

[4] Note in electric case we had path integral in time, now we have one in space. Not hard to guess that this generalizes to path in spacetime.

$$
\begin{aligned}
&= \frac{-e}{\hbar}\int_{\text{area encl.}} d\mathbf{a}\cdot\mathbf{B} && (26)\\
&= \frac{-e}{\hbar}\Phi \text{ (flux thru solenoid)} && (27)
\end{aligned}
$$

So phase difference is observable, & related to *gauge invariant* quantity, magnetic flux through solenoid.

AB effect measured many times, starting with Chambers[5] and Furry and Ramsey[6] The latter authors used a long magnetic whisker in place of a solenoid. The search for AB effects in mesoscopic systems (small semiconductor devices) led to the discovery of *weak localization* in disordered metals. Recently an effect analogous to the AB effect, but for neutral particles, was predicted by Aharonov and Casher[7] and has also been measured.[8]

2.4 Landau Levels

Consider 2D electron system in $x-y$ plane with field $\mathbf{B} \parallel \hat{z}$. Convenient to choose "Landau gauge" $\mathbf{A} = Bx\hat{y}$, check that $\mathbf{B} = \nabla\times\mathbf{A} = B\hat{z}$. With this choice Hamiltonian is (convention: electron has charge -e)

$$
\begin{aligned}
H &= \frac{1}{2m}\left(\hat{\mathbf{p}} + e\mathbf{A}\right)^2 && (28)\\
&= \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + 2eBx\hat{p}_y + (eB)^2 x^2\right) && (29)
\end{aligned}
$$

Note that $[H, \hat{p}_y] = 0$, so we may write all eigenfctns. of H as eigenfctns of $\hat{p}_y$, namely

$$\psi(x,y) = e^{ik_y y}X(x) \qquad (30)$$

Substitute, find X satisfies

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[5] R.G. Chambers, Physical Review Letters 5, 3 (1960).

[6] W.H. Furry and N.F. Ramsey, Physical Review 118, 623 (1960)

[7] Y. Aharonov and A. Casher, Physical Review Letters 53,.319 (1984).

[8] I don't know the reference here–anybody?

$$\frac{1}{2m}\left(-\hbar^2\nabla^2+(eB)^2\left(x+\frac{\hbar k_y}{eB}\right)^2\right)X=EX. \tag{31}$$

Note this eqn. is exactly of harmonic oscillator form, with x shifted by $x_0=\hbar k_y/eB$. So we can immediately write down the eigensolns for this problem:

$$\psi(x,y)=e^{ik_yy}u_n(x+x_0)=e^{ieBx_0y/\hbar}u_n(x+x_0), \tag{32}$$

where u_n is nth eigenfctn. of the SHO, with eigenvalue $E_n=\hbar\omega(n+1/2)$, and ω can be read off by comparison with standard SHO potential $m\omega^2x^2/2$, to find

$$\omega=\frac{eB}{m}\quad\text{(cyclotron frequency)}. \tag{33}$$

This is just classical frequency of orbital motion of chged. particle in magnetic field. Energy levels labeled by n called *Landau levels* because Landau solved this problem 1st (see his QM book!). What is degeneracy of each level? Note we can have many different $k_y's$ all with same E_n. If width of system in y-direction is L_y, assume periodic boundary conditions, $\psi(y)=\psi(y+L_y)$, then allowed k_y's are given by $k_yL_y=2\pi\nu$, $\nu=0,1,2,3\ldots$. May also translate condition into one on x_0, classical center of electron orbit, $x_0=2\pi\nu\hbar/(eBL_y)$. Note must have

$$0\le x_0\le L_x, \tag{34}$$

where L_x is width of sample in x-direction, so that all e^-'s are orbiting inside sample. This gives upper bound on ν,

$$0\le\nu\le\frac{eB}{2\pi\hbar}L_xL_y\equiv\nu_{max} \tag{35}$$

Natural unit of length $\simeq$ size of orbit

$$\ell_B=\sqrt{\frac{\hbar}{eB}} \tag{36}$$

So maximum number of electrons which can occupy given Landau level is

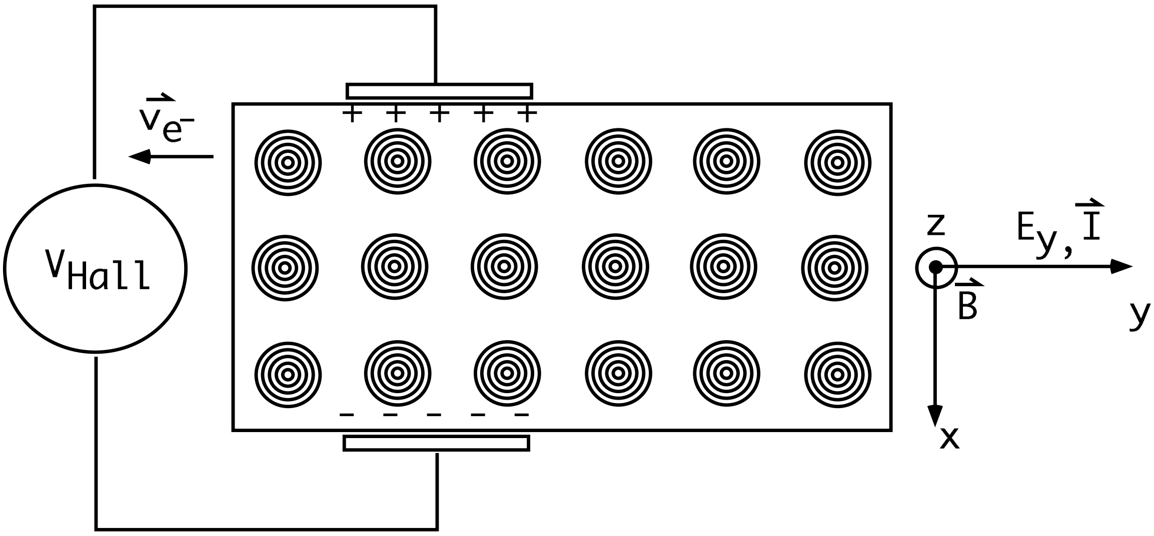
$$\nu_{max} = \frac{L_x L_y}{2\pi \ell_B^2}. \tag{37}$$

. Remarks:

1. Note ν_{max} depends on field: bigger field, more electrons can be fit into each Landau level.
2. Landau levels split by spin Zeeman coupling, so (37) applies to one spin only.
3. Although we treated x and y asymmetrically for convenience of calculation, no physical quantity should differentiate between the two due to symmetry of original problem with field in z direction!

2.5 Integer Quantum Hall Effect

Now take 2D electron system (can be really manufactured in semiconductor heterostructures!) and apply **E**-field in y direction as shown. *Longitudinal* current density proportional to applied E-field E_y (Ohm's law):



$$j_y = \sigma_0 E_y \tag{38}$$

Classically, electron trajectories curved by Lorentz force $\mathbf{F} = -e\mathbf{v} \times \mathbf{B}$, can think of as extra electric field

$$\mathbf{E}' = \mathbf{v} \times \mathbf{B} = \frac{-\mathbf{j} \times \mathbf{B}}{ne} \tag{39}$$

where I used $\mathbf{j} = -ne\mathbf{v}$. Total current is $\mathbf{j} = \sigma_0(E_y\hat{y} + \mathbf{E}')$, or

$$\mathbf{j} = \sigma_0\mathbf{E} - \sigma_0\frac{\mathbf{j} \times \mathbf{B}}{ne} \tag{40}$$

which may be written

$$\begin{pmatrix} 1 & \frac{\sigma_0 B}{ne} \\ -\frac{\sigma_0 B}{ne} & 1 \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \sigma_0 \begin{pmatrix} 0 \\ E_y \end{pmatrix} \tag{41}$$

which can be inverted to find

$$j_x = \underbrace{\frac{\frac{-\sigma_0^2 B}{ne}}{1 + (\frac{\sigma_0 B}{ne})^2}}_{\sigma_{xy}} E_y \tag{42}$$

$$\text{and} \quad j_y = \underbrace{\frac{\sigma_0}{1 + (\frac{\sigma_0 B}{ne})^2}}_{\sigma_{yy}} E_y \quad . \tag{43}$$

These are classical expressions for the longitudinal and Hall conductivities, σ_{xy} and σ_{yy} in $\perp$ field B. Note classical proportionality of σ_{xy} and B!

How does QM change this picture? Assumption of classical *Drude model* is that electrons scatter randomly & elastically off imperfections, leading to constant drift velocity in presence of electric field. This argument yields

$$\sigma_0 = \frac{ne^2\tau}{m}, \tag{44}$$

where τ is mean time between collisions, n is number density of e^-'s.

Now suppose the magnetic field is chosen so that number of electrons exactly fills all the Landau levels up to some N, i.e.

$$nL_xL_y = N\nu_{max} \Longrightarrow n = N\frac{eB}{h}, \tag{45}$$

where last step follows from Eqs. (35-37).[9] (Ask now what happens if level is filled. If electron scatters off imperfection, must go into another quantum state. But all such states of the same energy are filled, so elastic scattering impossible. Inelastic scattering "frozen out", i.e. next accessible Landau level a finite energy $\hbar\omega$ away, at low T thermal energy not enough to jump this gap. So no scattering can occur, due to Pauli principle! This means $\tau \to \infty$ at special values of field, so comparing with (42-43) find $\sigma_{yy} \to 0$, and

$$\sigma_{xy} \to \frac{ne}{B} = N\frac{e^2}{h} \tag{46}$$

At critical values of field, conductivity is *quantized*[10] units of e^2/h. Expt'l measurement of these values provides best determination of fundamental ratio e^2/h, better than 1 part in 10^7. Nobel prize awarded 1985 to von Klitzing for this discovery.

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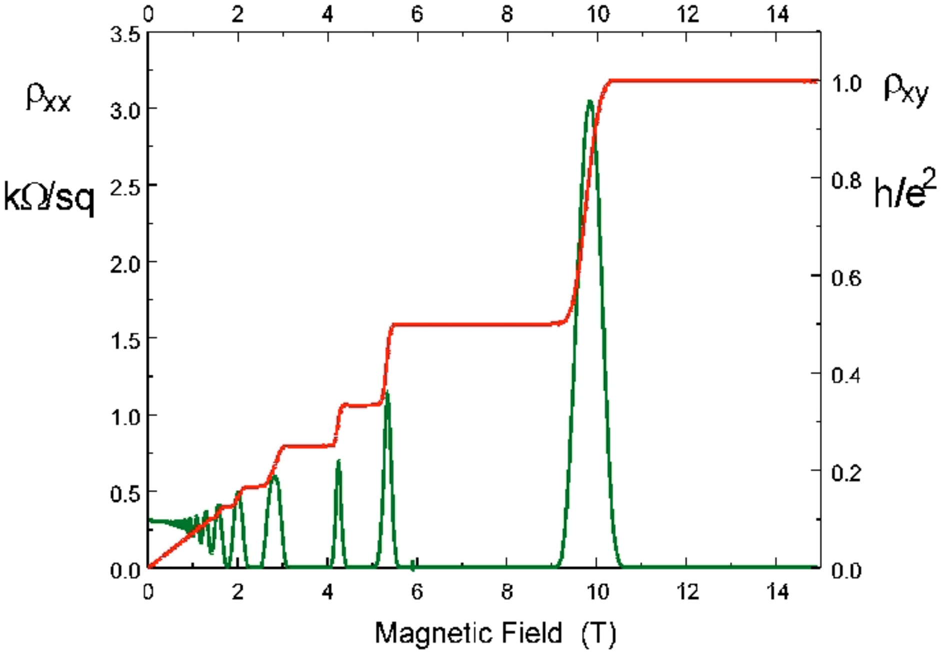
[9] Careful: n is the total number of electrons per unit area, ν_{max} is the number of electrons per Landau level, and N is the number of filled levels.

[10] This is the extent of the naive argument for IQHE. Note there is no discussion of what happens for fields just above or below critical fields. Understanding why σ_{xy} doesn't change over a nonzero range of B, i.e. existence of *quantum Hall plateaus*, crucial to understanding how effect can be measured at all. Ultimate explanation relies on effect of disorder on electronic wave functions, so-called *localization* effects.



ρxx
kΩ/sq
ρxy
h/e²
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
Magnetic Field (T)