PHZ3113–Introduction to Theoretical Physics

Fall 2008

Problem Set 15 Solutions

Nov. 23, 2008

1. For $n \geq 0$,

$$\oint_C (z - z_0)^n dz = 0 \tag{1}$$

because $(z - z_0)^n$ is analytic everywhere. For $n < 0$, we can use the Cauchy integral and its derivatives:

$$f^k(z_0) = \frac{k!}{2\pi i} \oint \frac{f(z)dz}{(z - z_0)^{k+1}} \tag{2}$$

rewrite for $f(z) = 1$, set $k + 1 = -n$, then we have the integral we want. Then

$$\oint_C (z - z_0)^n dz = \frac{2\pi i}{(-n-1)!} f^{-(1+n)}(z_0) = 0 \quad \text{for } n < -1 \tag{3}$$

since derivative of 1, a const., with respect to z_0 is 0, and if $n = -1$, then

$$\oint \frac{dz}{z - z_0} = 2\pi i f(z_0) = 2\pi i \tag{4}$$

2. From previous problem, with $z_0 = 0$, follows immediately. Only get nonzero answer if $m - n - 1 = -1$, i.e. if $m = n$, when result is $\frac{2\pi i}{2\pi i} = 1$.

3. Write

$$\frac{1}{z^2 - 1} = \frac{1}{z - 1} - \frac{1}{z + 1}, \tag{5}$$

so

$$\oint_C \frac{dz}{z^2 - 1} = \oint_C \frac{dz}{z - 1} - \oint \frac{dz}{z + 1} = 2\pi i (f(1) - f(-1)) = 0 \tag{6}$$

since $f(z) = 1$.

4. Using $e^z = 1 + z + z^2/2! + \ldots$, we have

$$\frac{1}{e^z - 1} = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots} = \frac{1}{z} \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots} \simeq \frac{1}{z}\left(1 - \frac{z}{2} + \frac{z^2}{12} \ldots\right) \tag{7}$$

So $b_1 = 1$, $a_0 = -\frac{1}{2}$, $a_1 = \frac{1}{12}$.

5. (a)

$$\frac{1}{z^2 + a^2} = \frac{1}{(z + ia)(z - ia)} \tag{8}$$

Simple poles at $z_0 = \pm ia \Rightarrow$

$$R(\pm ia) = \left.\frac{1}{z \pm ia}\right|_{\pm ia} = \pm \frac{1}{2ia} \tag{9}$$

(b)

$$\frac{z^2}{(z^2+a^2)^2} = \frac{z^2}{(z+ia)^2(z-ia)^2} \tag{10}$$

2nd order poles at $z_0 = \pm ia$.

$$R(\pm ia) = \frac{d}{dz}\frac{z^2}{(z+ia)^2} = -2\frac{z^2}{(z \pm ia)^3} + \frac{2z}{(z \pm ia)^2}\bigg|_{z=\pm ia} = \pm\frac{1}{4ai} \tag{11}$$

(c)

$$\frac{\sin 1/z}{z^2+a^2} = \frac{\sin \frac{1}{z}}{(z+ia)(z-ia)} \tag{12}$$

Simple poles at $z_0 = \pm ia$ and essential singularity at $z = 0$. $\Rightarrow$

$$R(\pm ia) = \frac{\sin 1/ai}{2ia} = -\frac{\sinh(1/a)}{2a} \tag{13}$$

For essential singularity expand

$$\sin\frac{1}{z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{z^n} \tag{14}$$

converges everywhere except at $z = 0$. The coefficient of $1/z$ is 1. The b_1 term in Laurent series will come from identifying the $1/z$ term in this series multiplied by $\frac{1}{z^2+a^2} \simeq \frac{1}{a^2}(1 - \frac{z^2}{z^2} + \dots)$, meaning that the residue is $1/a^2$.

(d)

$$\frac{ze^{iz}}{z^2+a^2} = \frac{e^{iz}}{(z+ia)(z-ia)} \tag{15}$$

simple poles $z_0 = \pm ia \Rightarrow R(\pm ia) = \frac{1}{2}e^{\mp a}$.