1. (a) 1; (b) 0 ;

   (c) Ans. $= 1/2$

   note

$$
\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \prod_{k=2}^{n}\left(\frac{k^2-1}{k^2}\right) = \prod_{k=2}^{n}\left(\frac{(k-1)(k+1)}{k^2}\right)
$$

$$
= \frac{1\cdot 3}{2\cdot 2}\frac{2\cdot 4}{3\cdot 3}\frac{3\cdot 5}{4\cdot 4}\cdots\frac{(n-2)n}{(n-1)(n-1)}\frac{(n-1)(n+1)}{n\,n} \tag{1}
$$

$$
= \frac{(n+1)}{2n} \to \frac{1}{2} \tag{2}
$$

   (d) As $x \to \infty$,

$$
\sqrt{x(x+a)} = x\sqrt{1+x/a} \simeq x(1+\frac{a}{2x}-\frac{1}{8}a^2\frac{1}{x^2}\ldots) \tag{3}
$$

   so

$$
x\sqrt{1+x/a}-x \to \frac{a}{2} \tag{4}
$$

2. • Terms are monotonically decreasing and positive. Integral test: $\int_0^\infty \frac{n^2}{2^n}dn = 2/\ln^3 2$, i.e. convergent.

   • We know $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, and $|\sin n| < 1$. Therefore by comparison test, $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ also converges.

   • Integral test. $\int_2^\infty 1/\ln^2 n\,dn = \infty$, hence series also is divergent.

   • If you factor out an overall minus sign, terms are positive and monotonically decreasing again, so integral test gives

$$
\int_2^\infty\left(\frac{1}{n}-\frac{1}{n-1}\right)dn = -\int_2^\infty\frac{1}{n(n-1)}dn = -\ln 2, \tag{5}
$$

   so it converges.

3. (a) $6+\frac{x-36}{12}-\frac{(x-36)^2}{1728}+\frac{(x-36)^3}{124416}+O\left((x-36)^4\right)$. Taylor series 3rd order at $x = 36.1$ gives 6.00833 to accuracy of my calculator, very good because exact result is 6.00833.

   (b) $1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3+O\left(\left(x-\frac{\pi}{4}\right)^4\right)$ Taylor series 3rd order at $x = Pi/4 + 0.1$ gives 1.22267, exact is 1.22305.

4. (a)

$$\langle \epsilon \rangle = \epsilon_0 \frac{\sum_n n e^{-nx}}{\sum_n e^{-nx}} = \epsilon_0 \frac{-\frac{\partial}{\partial x} \sum_n e^{-nx}}{\sum_n e^{-nx}}, \tag{6}$$

where $x = \epsilon_0/kT$. Now $\sum_n e^{-nx}$ is just a geometric series,

$$\sum_n e^{-nx} = 1 + e^{-x} + (e^{-}x)^2 + \ldots = \frac{1}{1 - e^{-x}}, \text{ so } \frac{\partial}{\partial x} \frac{1}{1 - e^{-x}} = \frac{e^{-x}}{(1 - e^{-x})^2}, \tag{7}$$

so

$$\langle \epsilon \rangle = \epsilon_0 \frac{e^{-x}}{1 - e^{-x}} = \epsilon_0 \frac{1}{e^x - 1} = \epsilon_0 \frac{1}{e^{\epsilon_0/kT} - 1}, \tag{8}$$

(The function $\frac{1}{e^{\epsilon_0/kT} - 1}$ is the Bose-Einstein distribution function, the probability of occupation of a state with energy ϵ_0 by particles with Bose statistics.)

(b) $kT \gg \epsilon_0 \quad \Rightarrow \quad x \ll 1$

$$\frac{\epsilon_0}{e^x - 1} \sim \frac{\epsilon_0}{(1 - x) - 1} = kT \tag{9}$$

5. The example we discussed in class, the vanishing of the electric and gravitational fields inside hollow uniform spheres of charge and mass, respectively, were special cases where only one sign of the mass (of course) or charge was relevant. The result that the electric field in a hollow conductor, or "Faraday cage", is zero, depends obviously on the redistribution of charges (both + and -) to cancel the electric field inside the conductor. There is no such redistribution of masses (no "screening" of gravity fields) possible because we only have + mass.