PHZ3113–Introduction to Theoretical Physics

Fall 2008

Problem Set 2 Solutions

Monday Sept. 15, 2008

1. (a) As $x \to \infty$, applying L'Hospital we find

$$\frac{x^n}{e^x} \to \frac{nx^{n-1}}{e^x} \to \ldots ... \frac{\text{const}}{e^x} \to 0 \tag{1}$$

(b) similarly

$$\frac{\ln x}{x^p} \to \frac{1}{px^p}. \tag{2}$$

If $p > 0$ this $\to 0$.

2. Take derivative wrt a of $\int_0^\infty e^{-ax} m \sin kx = \frac{k}{a^2+k^2}$ to find

$$-\int_0^\infty xe^{-ax} \sin kx = -\frac{2ak}{(a^2+k^2)^2}. \tag{3}$$

So $\int_0^\infty xe^{-ax} \sin kx dx = \frac{2ak}{(a^2+k^2)^2}$. Differentiate wrt k to obtain

$$\int_0^\infty e^{-ax} x \cos kx = \frac{1}{a^2+k^2} - \frac{2k^2}{(a^2+k^2)^2} = \frac{a^2-k^2}{(a^2+k^2)^2} \tag{4}$$

3. First let's calculate the total differential

$$dr = e^{-p^2-q^2}(-2pdp - 2qdq) \tag{5}$$

as well as the differentials $dp = e^s ds$ and $dq = -e^{-s} ds$. Substitute

$$dr \;=\; e^{-p^2-q^2}(-2pe^s + 2qe^{-s})ds \tag{6}$$

$$\Rightarrow \quad \frac{dr}{ds} = 2e^{-p^2-q^2}(e^{-2s} - e^{2s}) \tag{7}$$

4. (a) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} e^y \sin x = -e^y \cos x$, while $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} - e^y \cos x = -e^y \cos x$, so$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y}$ as must be true for any smooth function.

(b)

$$\frac{\partial^2 u}{\partial x^2} = -e^y \sin x \text{ and } \frac{\partial^2 u}{\partial y^2} = e^y \sin x \tag{8}$$

so indeed the sum $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for this case.

5. (a) First, solve for y explicitly:

$$y = \frac{1}{2}\left(\frac{\pm\sqrt{12x^2+x+20}}{\sqrt{x}}+1\right) \tag{9}$$

so derivative is

$$\frac{dy}{dx} = \pm\frac{5-3x^2}{x^{3/2}\sqrt{12x^2+x+20}} \tag{10}$$

Now do implicitly:

$$dx\, y^2 + 2xy\, dy - 6x\, dx = y\, dx + x\, dy \tag{11}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y+6x-y^2}{2xy-x}. \tag{12}$$

To check that this agrees with the explicit method, substitute back in for y as above:

$$\frac{dy}{dx} = \frac{\frac{1}{2}\left(\frac{\pm\sqrt{12x^2+x+20}}{\sqrt{x}}+1\right)+6x-\left(\frac{1}{2}\left(\frac{\pm\sqrt{12x^2+x+20}}{\sqrt{x}}+1\right)\right)^2}{2x\left(\frac{1}{2}\left(\frac{\pm\sqrt{12x^2+x+20}}{\sqrt{x}}+1\right)\right)-x} \tag{13}$$

$$= \pm\frac{5-3x^2}{x^{3/2}\sqrt{12x^2+x+20}} \tag{14}$$

it works!

(b) $x = \frac{3y-4}{y+2}$ Solution is $y = -\frac{2(x+2)}{x-3}$, so $dy/dx = \frac{10}{(x-3)^2}$. To take the implicit derivative, we find the differential

$$(y+2)\, dx + x\, dy = 3\, dy \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y+2}{x-3}. \tag{15}$$

Check again by substituting for y:

$$\frac{dy}{dx} = -\frac{2-\frac{2(x+2)}{x-3}}{x-3} = \frac{10}{(x-3)^2} \quad \text{OK} \tag{16}$$