PHZ3113–Introduction to Theoretical Physics

Fall 2008

Problem Set 3 Solutions

Sept 18, 2008

1. Applying the chain rule twice,

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ct) + g''(x+ct) = u'' \tag{1}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2(f''(x-ct) + g''(x+ct)) = c^2 u'', \tag{2}$$

so the equation given follows. As discussed in class, the general form is the superposition (addition) of a "pulse" of general form f travelling to the right and g travelling to the left, and the equation itself is the so-called wave equation in 1D, describing, e.g. waves on a string.

2. Total differential of $s(v,T)$ is

$$ds = \left.\frac{\partial s}{\partial v}\right|_T dv + \left.\frac{\partial s}{\partial T}\right|_V dT \tag{3}$$

$$= \left.\frac{\partial s}{\partial v}\right|_T dv + \frac{c_v}{T} dT. \tag{4}$$

We need to find a way to include derivatives wrt p. We are given $v = v(p,T)$ so let's use that. Express the differential of v

$$dv = \left.\frac{\partial v}{\partial p}\right|_T dp + \left.\frac{\partial v}{\partial T}\right|_p dT \tag{5}$$

and combine with the Eq. (4) to get

$$ds = \left.\frac{\partial s}{\partial v}\right|_T \left(\left.\frac{\partial v}{\partial p}\right|_T dp + \left.\frac{\partial v}{\partial T}\right|_p dT \right) + \frac{c_v}{T} dT \tag{6}$$

$$= \left.\frac{\partial s}{\partial v}\right|_T \left.\frac{\partial v}{\partial p}\right|_T dp + \left(\left.\frac{\partial s}{\partial v}\right|_T \left.\frac{\partial v}{\partial T}\right|_p + \frac{c_v}{T} \right) dT. \tag{7}$$

Now compare this with the expression for the exact differential of $s(p,T)$:

$$ds = \left.\frac{\partial s}{\partial p}\right|_T dp + \left.\frac{\partial s}{\partial T}\right|_p dT \tag{8}$$

$$\equiv \left.\frac{\partial s}{\partial p}\right|_T dp + \frac{c_p}{T} dT \tag{9}$$

and now equate the coefficients of the independent differential dT in both (7) and (9) to get the final result

$$c_p - c_v = T \left(\frac{\partial s}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_p . \tag{10}$$

3. We're looking for the point (x, y) where the distance from the origin $\sqrt{x^2 + y^2}$ is minimal subject to the constraint $x^2 - 2\sqrt{3}xy - y^2 = 2$. Note that if $x^2 + y^2$ is a minimum $\sqrt{x^2 + y^2}$ will be too. You can do this problem with the method of Lagrange multipliers described in Boas ch. 4, or by substituting the solution for y in terms of x,

$$y = -\sqrt{3}x - \pm\sqrt{2}\sqrt{2x^2 - 1} \tag{11}$$

into $x^2 + y^2$. One then needs to minimize $x^2 + y(x)^2$, or find the x such that

$$2x + 2\left(\pm\frac{2\sqrt{2}x}{\sqrt{2x^2 - 1}} - \sqrt{3}\right)\left(-\sqrt{3}x \pm \sqrt{2}\sqrt{2x^2 - 1}\right) = 0 \tag{12}$$

This can be simplified and solved for x, then the value for x substituted back to find y. The two solutions are $(x, y) = (\mp\sqrt{3}/2, \pm 1/2)$.

4. Here's a plot of the rectangle We want the temperature to be an extremum both

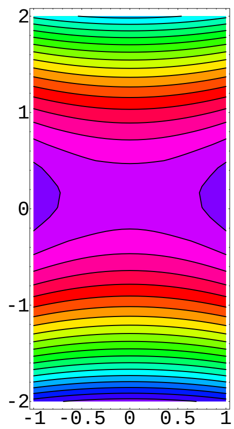

Figure 1: Temperature in rectangle. Note dark blue is cold and violet is hot.

along x and along y, i.e. the gradient $\nabla T = 0$ at these points.

$$\nabla T = (2x, 1 - 8y) = 0. \tag{13}$$

Now remember in 2D the stationary point $(0, 1/8)$ which solves these equations need not be an absolute min or max. To check we need the second derivatives

$$\frac{\partial^2 T}{\partial x^2} = 2\ ; \quad \frac{\partial^2 T}{\partial x \partial y} = 0\ ; \quad \frac{\partial^2 T}{\partial y^2} = -8, \tag{14}$$

so the signs of the curvature in x and y directions are different, indicating a saddle point. In the picture you can see this saddle point pretty clearly if you view it in color.

Now if there is no absolute min/max in the interior of the rectangle, it must take place on the boundaries. On $x = -1$, $T(-1, y) = -4 + y - 4y^2$, and the extrema are at $dT(-1,y)/dy = 0$ or $y = 1/8$ which has second derivative $d^2T(-1,y)/dy^2|_{y=1/8} = -8$, indicating a max. The "temperature" at this point is $T(-1,1/8) = -63/16$. There's another equivalent maximum at $(1,1/8)$.

Now look on top and bottom. $T(x,-2) = -23 + x^2$ so $dT(x,2)/dx = 0$ gives $2x = 0$ and $d^2T(x,2)/dx^2|_{x=0} = 2$, so a min wrt x. At $(0,-2)$ the temperature is $T = -23$. A similar analysis shows that there's a min on the top too, but the one on the bottom is lower.

5. Want to use the substitution $x = e^z$ and transform the differential equation

$$x^2\left(\frac{d^2y}{dx^2}\right) + 2x\left(\frac{dy}{dx}\right) - 5y = 0 \tag{15}$$

The chain rule gives

$$\frac{dy(x(z))}{dz} = \frac{dy}{dx}\frac{dx}{dz} = e^z\frac{dy}{dx} = x\frac{dy}{dx} \tag{16}$$

$$\frac{d^2y}{dz^2} = x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} \tag{17}$$

so you can see that by adding the two you get on the right hand side the first two terms of the original differential equation, which can thus be expressed as

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} - 5y = 0 \tag{18}$$