PHZ3113–Introduction to Theoretical Physics

Fall 2008

Problem Set 4 Solutions

Sept 22, 2008

1. Consider $F = F(r, \theta)$. Then the differential is

$$dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \theta} d\theta, \tag{1}$$

but we may consider r and θ to be functions of x and y. Therefore their differentials are

$$dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \tag{2}$$

$$d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy. \tag{3}$$

Substituting, we find

$$\left(\frac{\partial F}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial x} \right) dx + \left(\frac{\partial F}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial y} \right) dy. \tag{4}$$

If we now wish to take a derivative of F wrt x holding y constant, we can calculate it by simply setting $dy = 0$ in the differential, obtaining

$$\left. \frac{\partial F}{\partial x} \right|_y = \frac{\partial F}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial x}, \tag{5}$$

just the result expected from the chain rule. The other derivatives requested follow from a similar analysis.

2. We're given $du = Tds - pdv$.

   (a) (T, v). Note that

$$d(u - Ts) = Tds - pdv - Tds - sdT = -pdv - sdT, \tag{6}$$

   so $f = u - Ts$ is a function whose differential depends only on dv and dT.

   (b) (s, p). Consider

$$d(u + pv) = Tds - pdv + pdv + vdp = Tds - vdp, \tag{7}$$

   so $h = u + pv$ has a differential which depends only on ds and dp.

   (c) (p, T). Finally, consider (using results of (b))

$$d(h - Ts) = Tds - vdp - Tds - sdT = -vdp - sdT, \tag{8}$$

   so $g = h - Ts = u + pv - Ts$ has a differential which depends only on dp and dT. Note in thermodynamics the functionals f, h, and g are referred to as the Helmholtz free energy, the enthalpy, and the Gibbs free energy. They represent quantities which are extremal when a thermodynamic system is in equilibrium under conditions when one of the relevant thermodynamic variables is held constant.

3. Start with $du = Tds - pdv$. Think of u as a function of s and v. Since this is an exact differential, we know that

$$T = \left.\frac{\partial u}{\partial s}\right|_V \quad ; \qquad -p = \left.\frac{\partial u}{\partial V}\right|_s . \tag{9}$$

We can take a v derivative of the first and an s derivative of the second to obtain $\frac{\partial^2 s}{\partial v \partial s}$ and $\frac{\partial^2 s}{\partial s \partial v}$. But the equality of mixed partial derivatives implies that

$$\left.\frac{\partial T}{\partial V}\right|_s = -\left.\frac{\partial p}{\partial s}\right|_V , \tag{10}$$

one of the Maxwell relations.

4. Proceeding similarly to Prob.3,

(a)

$$dh = Tds + vdp \quad \Rightarrow \quad \left.\frac{\partial T}{\partial p}\right|_s = \left.\frac{\partial V}{\partial s}\right|_p \tag{11}$$

(b)

$$df = -pdv - sdT \quad \Rightarrow \quad \left.\frac{\partial p}{\partial T}\right|_v = \left.\frac{\partial s}{\partial v}\right|_T \tag{12}$$

(c)

$$dg = vdp - sdT \quad \Rightarrow \quad \left.\frac{\partial V}{\partial T}\right|_p = -\left.\frac{\partial s}{\partial p}\right|_T , \tag{13}$$

we get the remaining Maxwell relations.

5. (a) σ is a surface charge density, or charge/area. Since xyz is a volume, the constant a must have dimensions of charge/L^5.

(b) Let's use method of Lagrange multipliers (Boas p. 214 et seq.). Prescription is to define a new function which is the function to be minimized plus a multiplier λ times the constraint. So

$$F[x, y, z, \lambda] = axyz + \lambda(x^2 + y^2 + z^2 - b^2). \tag{14}$$

Let's now set all partial derivatives of F equal to zero:

$$\frac{\partial F}{\partial x} = ayz + 2\lambda x = 0 \quad ; \quad \frac{\partial F}{\partial y} = axz + 2\lambda y = 0 \;\; ; \quad \frac{\partial F}{\partial x} = axy + 2\lambda z = 0 \;\; ;$$
$$\frac{\partial F}{\partial \lambda} = x^2 + y^2 + z^2 - b^2 = 0. \tag{15}$$

Multiplying the first three equations by x, y and z respectively, and adding, we get

$$3axyz + 2\lambda(x^2 + y^2 + z^2) = 0 \quad \Rightarrow \quad axyz = -\frac{2}{3} b^2 \lambda. \tag{16}$$

Multiplying 1st equation in (15) by x and substituting for xyz, we arrive at $x^2 = \lambda/3$. Since the problem is symmetric in x, y and z, we can immediately say $x^2 = y^2 = z^2 = \lambda/3$. But this means that $x^2 + y^2 + z^2 = \lambda = b^2$, so we have determined the Lagrange multiplier, and can say that the extremum occurs at $x^2, y^2, z^2 = (b^2/3)(1, 1, 1)$.

Now we need to say whether these values are maxima or minima. Let's assume $a > 0$ without loss of generality. Since $\sigma = axyz$, two extremal values are obviously $\pm b^3 a/3^{3/2}$. So without taking second derivatives, we can say that the maxima, where $\sigma = b^3 a/3^{3/2}$, occur at $\sqrt{3}(x, y, z)/b = (1, -1, -1), (-1, 1, -1), (1, 1, 1)$ and $(-1, -1, 1)$.