PHZ3113–Introduction to Theoretical Physics

Fall 2008

Problem Set 5 Solutions

Sept 24, 2008

1. Applying the formula given in notes or Boas,

$$\frac{d}{dx}\int_1^{1/x}\frac{e^{xt}}{t}dt = -\frac{1}{x^2}\cdot e^1\cdot x + \int_1^{1/x} e^{xt}dt \quad (1)$$

$$= -\frac{e}{x}+\frac{1}{x}\int_x^1 e^y dy = -\frac{e}{x}+\frac{e}{x}-\frac{e^x}{x}=\frac{-e^x}{x} \quad (2)$$

where in the last step I made the transformation $y = xt$.

2. Total differential:

$$d\left[\left(p+\frac{a}{v^2}\right)(v-b)\right] = (v-b)d\left(p+\frac{a}{v^2}\right)+\left(p+\frac{a}{v^2}\right)d(v-b)$$

$$= \left[dp-\frac{2a}{v^3}dv\right](v-b)+\left(p+\frac{a}{v^2}\right)dv$$

$$= \left[p+\frac{a}{v^2}+(b-v)\frac{2a}{v^3}\right]dv-(b-v)dp. \quad (3)$$

So

$$\frac{dv}{dp}=\frac{b-v}{p-\frac{a}{v^2}+\frac{2ab}{v^3}} \quad (4)$$

3. General expression for a Taylor expansion in 2 variables is

$$f(x,y)=f(a,b)+\sum_{i=1}^{\infty}\frac{1}{i!}\left[\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^i f(x,y)\right]_{x=a,y=b} \quad (5)$$

(a)

$$\left(7+4(y-2)+(y-2)^2+O\left((y-2)^3\right)\right) \quad (6)$$

$$+\left(8+4(y-2)+(y-2)^2+O\left((y-2)^3\right)\right)(x-1) \quad (7)$$

$$+2(x-1)^2+O\left((x-1)^3\right) \quad (8)$$

(b)

$$\left(y^2+O\left(y^3\right)\right)+\left(-3y^2+O\left(y^3\right)\right)(x-1)+\left(6y^2+O\left(y^3\right)\right)(x-1)^2+O\left((x-1)^3\right) \quad (9)$$

4. Lagrange multiplier:

$$F(x,y;\lambda)\equiv x^2+y^2+\lambda(1-x^2-y) \quad (10)$$

so

$$\frac{\partial F}{\partial x} = 2x - \lambda 2x = 0 \;\; ; \quad \frac{\partial F}{\partial y} = 2y - \lambda = 0 \;\; ; \quad \frac{\partial F}{\partial \lambda} = 1 - x^2 - y = 0. \qquad (11)$$

These eqns. have 2 solutions $(0, 1)$ and $(\pm 1/\sqrt{2}, 1/2)$. The former is not an absolute minimum (plug it in), and may be shown to be in fact a maximum. Answer is therefore $(\pm 1/\sqrt{2}, 1/2)$.

5. (a)

$$S = \ln W = \ln N! - \sum_j \ln(n_j!) \qquad (12)$$

$$\ln n! = \sum_{m=0}^{n-1} \ln(n-m) \approx \int_0^{n-1} \ln(n-x)dx \qquad (13)$$

$$= \int_1^n dy \ln y = n \ln n - n + 1 \approx n \ln n - 1. \qquad (14)$$

(b) Constraints: i) number $\sum_j n_n = N$, $\quad \sum_j n_j \epsilon_j = E$.

(c)

$$F = S - \alpha \sum_j n_j - \beta \sum_j n_j \epsilon_j \qquad (15)$$

$$= N \ln N - N - \sum_j (n_j \ln n_j - n_j) \qquad (16)$$

$$-\alpha(\sum_j n_j - N) - \beta(\sum_j n_j \epsilon_j - E) \qquad (17)$$

$$\frac{\partial F}{\partial n_i} = 0 \quad \Rightarrow \quad -\ln n_i - \alpha - \beta \epsilon_i = 0. \qquad (18)$$

(d) $\ln n_i = -\alpha - \beta \epsilon_i \quad \Rightarrow \quad n_i = \exp(-\alpha - \beta \epsilon_i)$.