PHZ3113–Introduction to Theoretical Physics

Fall 2008

Problem Set 8 Solutions

Oct. 6, 2008

1.

$$\int \vec{\nabla}\cdot\vec{v} = 0. \tag{1}$$

So we need to show that the surface integral $\int \vec{v}\cdot d\vec{a}$ over the parallelepiped also vanishes. For the faces:

(a) $x_1 = 0$, $0 \le x_2 \le 3$, $0 \le x_3 \le 2$:

$$\begin{aligned} d\vec{a} &= dx_2 dx_3(-\hat{x}_1) \quad \vec{v}\cdot d\vec{a} = -x_2 dx_2 dx_3 \\ \Rightarrow &\quad \int_{A_1} \vec{v}\cdot d\vec{a} = -\int_0^3 x_2 dx_2 \int_0^2 dx_3 = -9 \end{aligned} \tag{2}$$

(b) $x_1 = 1$, $0 \le x_2 \le 3$, $0 \le x_3 \le 2$:

$$\begin{aligned} d\vec{a} &= dx_2 dx_3(\hat{x}_1) \quad \vec{v}\cdot d\vec{a} = x_2 dx_2 dx_3 \\ \Rightarrow &\quad \int_{A_2} \vec{v}\cdot d\vec{a} = \int_0^3 x_2 dx_2 \int_0^2 dx_3 = 9 \end{aligned} \tag{3}$$

(c) $x_2 = 0$, $0 \le x_1 \le 1$, $0 \le x_3 \le 2$:

$$\begin{aligned} d\vec{a} &= dx_1 dx_3(-\hat{x}_2) \quad \vec{v}\cdot d\vec{a} = 2dx_1 dx_3 \\ \Rightarrow &\quad \int_{A_3} \vec{v}\cdot d\vec{a} = 2\int_0^1 dx_1 \int_0^2 dx_3 = 4 \end{aligned} \tag{4}$$

(d) $x_2 = 3$, $0 \le x_1 \le 1$, $0 \le x_3 \le 2$:

$$\begin{aligned} d\vec{a} &= dx_1 dx_3(\hat{x}_2) \quad \vec{v}\cdot d\vec{a} = -2dx_1 dx_3 \\ \Rightarrow &\quad \int_{A_4} \vec{v}\cdot d\vec{a} = 2\int_0^1 dx_1 \int_0^2 dx_3 = -4 \end{aligned} \tag{5}$$

(e) $x_3 = 0$, $0 \le x_1 \le 1$, $0 \le x_2 \le 3$:

$$\begin{aligned} d\vec{a} &= dx_1 dx_2(-\hat{x}_3) \quad \vec{v}\cdot d\vec{a} = -x_1 dx_1 dx_2 \\ \Rightarrow &\quad \int_{A_5} \vec{v}\cdot d\vec{a} = \int_0^1 x_1\, dx_1 \int_0^3 dx_2 = -\frac{3}{2} \end{aligned} \tag{6}$$

(f) $x_3 = 3$, $0 \le x_1 \le 1$, $0 \le x_2 \le 3$:

$$\begin{aligned} d\vec{a} &= dx_1 dx_2(\hat{x}_3) \quad \vec{v}\cdot d\vec{a} = x_1 dx_1 dx_2 \\ \Rightarrow &\quad \int_{A_6} \vec{v}\cdot d\vec{a} = \int_0^1 x_1\, dx_1 \int_0^3 dx_2 = \frac{3}{2} \end{aligned} \tag{7}$$

... so the surface integral over the faces cancel pairwise.

2. (a) Basic proof is as follows:

$$\begin{aligned}\vec{\nabla} \times \vec{A} &= \vec{\nabla} \times \frac{1}{2}(\vec{B} \times \vec{r}) = \frac{1}{2}\vec{B}(\vec{\nabla} \cdot \vec{r}) - \frac{1}{2}(\vec{B} \cdot \vec{\nabla})\vec{r} \\ &= \frac{3}{2}\vec{B} - \frac{1}{2}\vec{B} = \vec{B} \end{aligned} \tag{8}$$

for any *constant* $\vec{B}$. In the 2nd step we could not have pulled $\vec{B}$ to the left of the differential operator unless it were constant,

$$\begin{aligned}(\vec{\nabla} \times (\vec{B} \times \vec{r})_i &= \epsilon_{ijk}\nabla_j(\vec{B} \times \vec{r})_k = \epsilon_{ijk}\nabla_j\epsilon_{k\ell m}B_\ell x_m \\ &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\nabla_j B_\ell x_m \\ &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})[(\nabla_j B_\ell)x_m + B_\ell(\nabla_j x_m)]. \end{aligned} \tag{9}$$

Only for constant $\vec{B}$ does the first term in the square brackets vanish, giving you (8).

(b)

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla}u \times \vec{\nabla}v) = \vec{\nabla}v \cdot (\vec{\nabla} \times \vec{\nabla}u) - \vec{\nabla}u \cdot (\vec{\nabla} \times \vec{\nabla}v) = 0, \tag{10}$$

where the first equality follows from the gradient version of the "BAC-CAB" identity (see vector identity sheet) and the second identity from $\vec{\nabla} \times \vec{\nabla}\phi = 0$.

(c)

$$\begin{aligned}\vec{\nabla} \times \vec{A} &= \frac{1}{2}\vec{\nabla} \times (u\vec{\nabla}v - v\vec{\nabla}u) \\ &= \frac{1}{2}[u\vec{\nabla} \times \vec{\nabla}v + (\vec{\nabla}u \times \vec{\nabla}v) - v\vec{\nabla} \times \vec{\nabla}u - (\vec{\nabla}v \times \vec{\nabla}u)] \\ &= \frac{1}{2}[0 + (\vec{\nabla}u \times \vec{\nabla}v) - 0 - (\vec{\nabla}v \times \vec{\nabla}u]) = (\vec{\nabla}u \times \vec{\nabla}v) = \vec{B}\end{aligned}$$

(d) Transformed magnetic field $\vec{B'} = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\psi) = \vec{B} + \vec{\nabla} \times \vec{\nabla}\psi = 0$. So the left hand side of Stokes' law is invariant. The extra term on the right hand side generated by the transformation is

$$\oint \vec{\nabla}\psi \cdot d\vec{r} = 0, \tag{11}$$

since the integral is taken around a closed loop and the vector field being integrated around the loop is the gradient of a scalar (conservative).

3. Expand function $g(x)$ in the neigbhorhood of each point where its argument becomes zero; only at these points will the δ-function be significant:

$$\int f(x)\delta(g(x))dx = \sum_n \int_{a_n-\epsilon}^{a_n+\epsilon} f(x)\delta[g(a_n) + (x - a_n)g'(a_n)]dx, \tag{12}$$

where ϵ is a small quantity. Note also that $g(a_n) = 0$ by definition. Since we know that $\delta(ax) = \frac{1}{|a|}\delta(x)$, we may write

$$= \sum_n \int dx f(x) \frac{1}{|g'(a_n)|} \delta(x - a_n) \tag{13}$$

as desired. Note this is *not* an approximation, since the δ function is exactly zero away from the zeros of its argument.

4. Before proceeding, note that the argument of the δ function has 2 roots, but only one of them is within the range of integration, $x = 1/2$. So

$$\begin{aligned} \int_0^\infty (3x^2+5)\delta(2x^2+3x-2) &= \int (3x^2+5)\frac{\delta(x-1/2)}{|4\cdot\frac{1}{2}+3|} \\ &= \frac{1}{5}\int (3x^2+5)\delta(x-1/2) = \frac{1}{5}\cdot\frac{23}{4} = \frac{23}{20} \end{aligned}$$

5. Pt. charge at $\vec{R}$: $\rho(\vec{r}) = q\delta^{(3)}(\vec{r}-\vec{R})$. Shell $\rho(\vec{r}) = \sigma\delta(r-a)$, where $\sigma = q/(4\pi a^2)$. Note the prefactor of the latter is fixed by the fact that

$$\int d\tau \rho(r) = q \tag{14}$$

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