12 Analytic functions

Read: Boas Ch. 14.

12.1 Analytic functions of a complex variable

Def.: A function $f(z)$ is *analytic* at z if it has a derivative there

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{1}$$

which exists *and is independent of the path by which one lets* $\Delta z \to 0$.

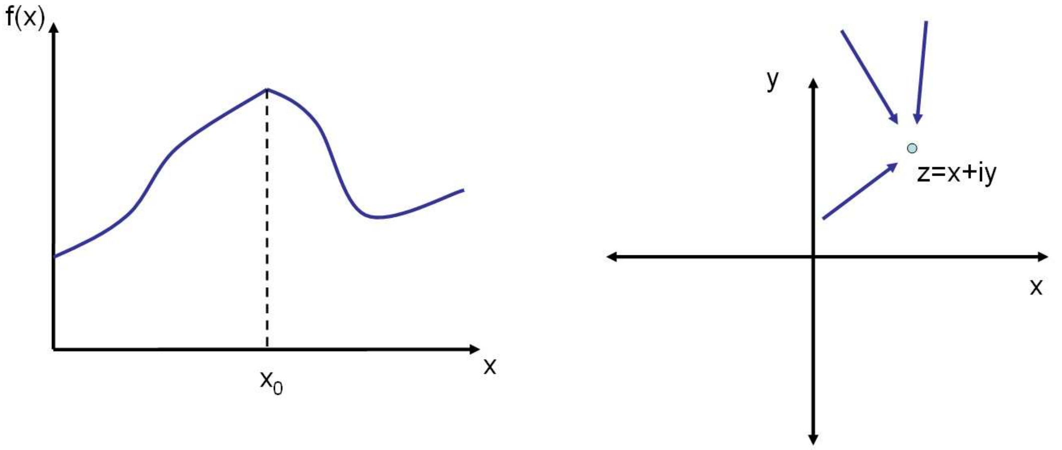

Figure 1: Left: function of 1 real variable. Derivative does not exist at x_0 because limit $(f(x+\Delta x) - f(x))/\Delta x$ is different from left or from right. Right: possible ways to approach z in the complex plane.

To clarify the importance of the path independence, consider the complex function $f(z) = |z|^2$. This looks smooth enough, since we can write it as $f(x,y) = x^2 + y^2$. But considered as a complex function it is not analytic, as we can see by applying the definition

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &\equiv \frac{(z+\Delta z)(z^* + \Delta z^*) - zz^*}{\Delta z} = \frac{z\Delta z^* + \Delta z z^*}{\Delta z} \\ &= \frac{(x+iy)(\Delta x - i\Delta y) + (x-iy)(\Delta x + i\Delta y)}{\Delta x + i\Delta y} \\ &= \frac{2x\Delta x + 2y\Delta y}{\Delta x + i\Delta y} \end{aligned} \tag{2}$$

Consider now path 1 approaching z: $\Delta y = 0$, $\Delta x \to 0$. Then derivative $\to 2x$. On the other hand on path 2 $\Delta x = 0$, $\Delta y \to 0$, derivative $\to -2iy$. So this is *not* an analytic function at any z. In general simple functions of z itself, not $|z|$, have regions where they are analytic.

If a function is analytic and single valued within a given region, we call it "regular". If it is multivalued, there are places where the function is not analytic, called "branch cuts".

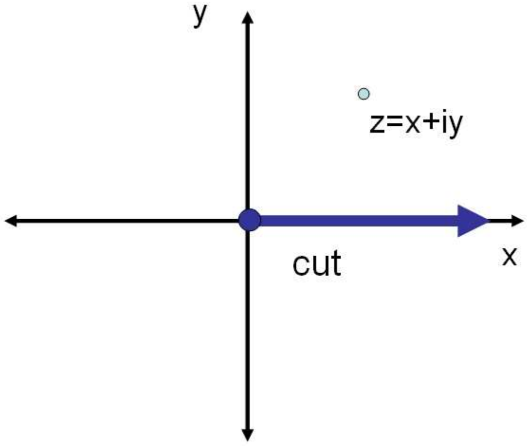

Figure 2: Branch cut of $w = z^{1/2}$

Ex.: consider the function $w = \sqrt{z} = \sqrt{r}e^{i\theta/2}$. w is a complex number, let's call it $w = \rho e^{i\phi}$. So we see that $\rho = r^{1/2}$ and $2\phi = \theta$. We can see that the mapping is not 1-1, since both ϕ and $\phi + \pi$, two different points in the w plane correspond to θ and $\theta + 2\pi$, i.e. the same z. We can *define* a new function w which is single valued by restricting the value of θ to lie between 0 and 2π. This is a new complex function which is identical to the first in this range of theta. If we specify a "branch cut" in the z plane as in Figure 2, the restriction of θ amounts to a statement that we never "cross" this when taking the square root.

Def.: Cauchy-Riemann conditions. Theorem: say $f(z) = u(z) + iv(z)$. A necessary and sufficient condition for f to be analytic is that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{3}$$

are satisfied.

- Check ex. 1: $f(z) = z^2$ so $u = x^2 - y^2$ and $v = 2xy$.

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y} \tag{4}$$

- ex. 2: $f(z) = z^*$. $u = x$ and $v = -y$.

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1 \quad \Rightarrow \quad \text{not analytic} \tag{5}$$

Sometimes we need the $C-R$ conditions in polar coords. Again $f(z) = u + iv$, $z = re^{i\theta}$.

$$\frac{\partial f}{\partial r} = \frac{df}{dz}\frac{\partial z}{\partial r} = \frac{\partial f}{\partial z}e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \tag{6}$$

$$\frac{\partial f}{\partial \theta} = \frac{df}{dz}\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial z}ire^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} \tag{7}$$

$$\Rightarrow e^{i\theta}\frac{df}{dz} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{ir}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) \tag{8}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \tag{9}$$

Remarks:

- If $f(z)$ is regular in a region R, derivatives of all orders exist there.
- A Taylor series is possible about any point $z_0 \in R$:

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$$
$$\text{with} \quad a_0 = f(z_0) \;\; ; \quad a_n = \frac{1}{n!}f^{(n)}(z_0). \tag{10}$$

  Region of convergence of series about z_0 is a circle of radius equal to the "distance" in the complex z-plane between z_0 and the nearest singularity.

- If $f(z)$ is regular in R, then u and v satisfy Laplace's equation, e.g. $\nabla^2 u = 0$! (They are called *harmonic functions*).

12.2 Line integrals and Cauchy's theorem

Integrals of analytic functions along paths have some spectacular properties, which we will try to prove here in a somewhat nonrigorous fashion. Let's first make a statement:

Claim: If $f(z)$ is analytic in a region R, the line integral

$$\int_C f(z)dz \tag{11}$$

along any contour C connecting z_1 and z_2 which does not leave this region is the same. In particular, we can *deform* the contour C shown, to new contour C',

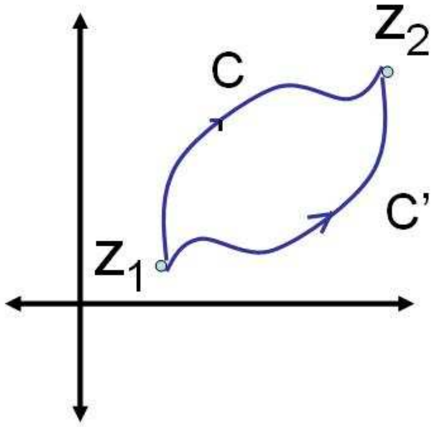

Figure 3: If $f(z)$ is analytic in a region, the line integral of f along any path between two points z_1 and z_2 is the same.

without changing the integral. Let's leave this as a claim for now, and prove a related proposition, which clearly follows from it. If we note that $\int_{C'(z_1,z_2)} dz = -\int_{C'(z_2,z_1)} dz$, it is easy to see that if we integrate from z_1 to z_2 along C, and come back to z_1 along C', we have made a closed loop. If the integrals along the two paths from z_1 to z_2 were originally the same, the integral around the closed loop must be zero. Therefore another property about a line integral of a function analytic in a given region is

$$\oint f(z)dz = 0 \tag{12}$$

Proof: Now call any closed curve in a region where $f(z) = u(x,y) + iv(x,y)$ is analytic C (see Fig. 4).

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy) = \underbrace{\oint_C (udx - vdy)}_{(1)} + i\underbrace{\oint vdx + udy}_{(2).} \tag{13, 14}$$

Now consider (1) and write the integrand as $\vec{F}\cdot d\vec{r}$, where $\vec{F} = u\hat{i} - v\hat{j} + 0\hat{k}$, and apply Stoke's theorem $\oint \vec{F}\cdot d\vec{r} = \int_A \vec{\nabla}\times\vec{F}\cdot d\vec{a}$, where A is the area enclosed by C. But you can easily calculate that

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ u & -v & 0 \end{vmatrix} = \partial_x(-v) - \partial_y u = 0, \tag{15}$$

where the last equality follows from the Cauchy-Riemann conditions. You can show by similar arguments that integral (2) is also zero.

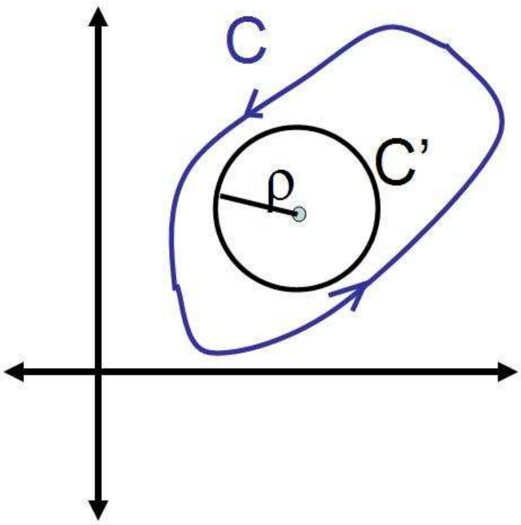

Figure 4: Start with arb. closed curve around a. Deform it into circle of radius ρ around a.

Now let's use the fact that in the above proof C was arbitrary to prove something extremely useful and spectacular: Cauchy's theorem, which says that if $f(z)$ is analytic

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz = f(a) \tag{16}$$

Note that the integrand here is *not* analytic everywhere: it has a singularity at $z = a$. Still, according to the discussion above, we are allowed to deform the contour through any region where the function we are integrating is analytic, without changing the value of the line integral. So it's ok to deform the contour from the arbitrary path C to, e.g. a circle C' surrounding a of radius ρ. On this circle, we can parameterize z as $z = a + \rho e^{i\theta}$, so $dz = \rho(id\theta)e^{i\theta}$, and

$$\oint_C \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(z)}{\rho e^{i\theta}}\rho i e^{i\theta}d\theta = i\int_0^{2\pi} d\theta f(z) \underset{\rho\to 0}{\longrightarrow} i\, f(a)\, 2\pi, \tag{17}$$

where in the last step I took the liberty of shrinking down the circle C', without ever allowing it to cross a. Since $f(z)$ is analytic in this region by assumption, it has a well defined limiting value at $z = a$ which does not (recall) depend on how we approach it, so we can say that the limit is $f(a)$.

Q: what if a is outside C to begin with?

Cauchy's theorem is extremely valuable to evaluate various types of integrals.

Generalization of Cauchy integral. Note that we can differentiate both sides of

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz = f(a) \tag{18}$$

with respect to a to get

$$f^n(a) = \frac{n!}{2\pi i} \oint \frac{f(z)dz}{(z-a)^{n+1}} \tag{19}$$

12.3 Examples

Ex. 1 On straight path $y = 1$ from $x = 0$ to 1:

$$\int_i^{1+i} zdz = \int_{(0,1)}^{(1,1)} (x+iy)(dx+idy) = \int_0^1 (x+i)dx = \frac{1}{2} + i \tag{20}$$

Ex. 2

$$\oint \frac{\sin z dz}{2z - \pi} = \frac{1}{2} \oint \frac{\sin z}{z - \frac{\pi}{2}}. \tag{21}$$

There are two cases.

- Case 1 $z = \pi/2$ is inside contour C . Then Cauchy integral may be used

$$\frac{1}{2} \oint \frac{\sin z\, dz}{z - \frac{\pi}{2}} = \frac{1}{2} 2\pi i \sin \frac{\pi}{2} = \pi i. \tag{22}$$

- Case 2 $z = \pi/2$ is outside contour C. Then C can be deformed and shrunk to zero since $\sin z/(z - \pi/2)$ is analytic everywhere inside.

Ex. 3 Consider for the contour C being the square path with corners $(x, y) = (\pm 1, \pm 1)$ centered at the origin

$$I = \oint_C \frac{e^{3z}}{(z - \ln 2)^4}. \tag{23}$$

Use (19) above, with $a = \ln 2 \simeq 0.693$ and $f = e^{3z}$. Then

$$I = \frac{2\pi i}{3!} \frac{d^3}{dz^3} \left(e^{3z}\right)_{z \to \ln 2} = 72\pi i. \tag{24}$$