13 Definite integrals

Read: Boas Ch. 14.

13.1 Laurent series:

Def.: Laurent series (LS). If $f(z)$ is analytic in a region R, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{1}$$

converges in R, with

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad ; \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{-n+1}} \tag{2}$$

If you recall, for an ordinary power series we need $x <$ const. $\equiv \frac{|a_{n+1}}{a_n|}$, by the ratio test, for convergence. Analog for complex numbers is $|z| <$ const$_2$. This applies then to the first part of the series. But a function may have some singularities outside the region R, so we need to account for them as well; hence the second part of the series. Here there may be a region of space where $1/|z| <$ const$_1$ is required for convergence. The figure shows that these regions can overlap, giving the annulus R where the function is analytic and both parts of the series converge. This depends of course on what the a_n and b_n are. How do you find them? Well,

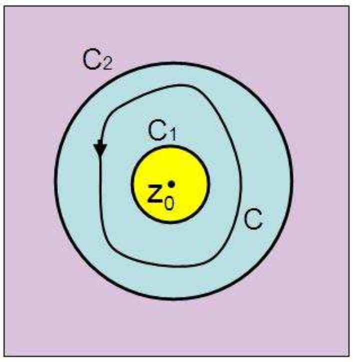

Figure 1: A Laurent series might converge only in an annulus R around a given point z_0, here between circles C_1 and C_2 in complex plane, because the "a-series" converges inside C_2 and the "b-series" converges outside C_1.

you can use Eq. (2), but this might involve doing rather a lot of integrals :-). In practice, we can often use "partial fraction" tricks, like the following.

Ex.:

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z} = -\frac{1}{z} - (1-z)^{-1} = -\frac{1}{z} - 1 - z - z^3 \ldots, \tag{3}$$

so you can just read off the coefficients. Note only one of the b_n is nonzero for this case.

If $f(z)$ is regular in the annulus, no matter how small we make C_1, and yet $f(z)$ is not regular everywhere inside C_2, we have a special situation of an *isolated singularity.* There are three subpossibilities:

1. LS may have *all* $b_n = 0$

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n, \qquad z \neq z_0 \tag{4}$$

   This is called a removable singularity. You may think in fact it's not a singularity at all, but take the example

$$f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 2 & z = 0 \end{cases} \tag{5}$$

   The function is certainly not analytic at $z_0 = 0$. However we can "remove" the singularity by redefining the function to have the value 1 at $z = 0$ ($\lim_{z\to 0} \sin z/z = 1$). Not a very interesting case, anyway.

2. LS may have $b_n \neq 0$ for $n < m$; $b_n = 0$ for $n \geq m$. This is called a *pole of order* m.

$$f(z) = \frac{z+3}{z^2(z-1)^3(z+1)}. \tag{6}$$

   This function has poles of order 1 at $z = -1$, of order 2 at $z = 0$, and of order 3 at $z = 1$.

   N.B. If $f(z)$ has a pole of order m at z_0, then $(z-z_0)^m f(z)$ is regular around z_0.

3. If there are an infinite number of nonzero b_n's, the point z_0 is called an *essential singularity.* Example is $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots$ at $z = 0$.

13.2 Residues

The coefficient b_1, coefficient of $\frac{1}{z-z_0}$ is special, and given the special name "the residue $R(z_0)$ of $f(z)$ at z_0". From our formula we have

$$b_1 = R(z_0) = \frac{1}{2\pi i} \oint_c f(z)dz \tag{7}$$

We can "prove" or check this, by integrating $f(z)$ around a closed contour

$$\begin{aligned} \oint_C f(z)dz &= \oint \left[\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \right] dz \\ &= \sum_{n=0}^{\infty} a_n \underbrace{\oint (z-z_0)^n dz}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\oint \frac{dz}{(z-z_0)^n}}_{0 \text{ if } m \neq 1,} \end{aligned} \tag{8}$$

where the first term vanishes because the function is analytic everywhere inside and on the closed contour C, and the second term you proved in the HW.

If there are several isolated singularities at z_0, z_1,...then Cauchy's theorem says that the line integral on a closed contour enclosing them is related to the sum of the residues:

$$\oint_C f(z)dz = 2\pi i \sum_i R(z_i) \tag{9}$$

where z_i are the singularities of f inside C.

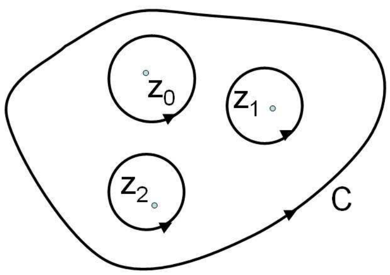

Figure 2: If there are many isolated singularities, the line integral about all of them is $2\pi i$ the sum of the residues at each.

13.2.1 Finding residues

- If $f(z)$ has a simple pole at $z = z_0$:

$$R(z_0) = (z - z_0)f(z)|_{z \to z_0} \tag{10}$$

Ex. 1:

$$f(z) = \frac{z}{(2z+1)(5-z)} \tag{11}$$

$$\begin{aligned} R(-\frac{1}{2}) &= (z + \frac{1}{2})f(z)\Big|_{z \to -\frac{1}{2}} = -\frac{1}{22} \\ R(5) &= (z-5)f(z)|_{z \to 0} = 1 \end{aligned} \tag{12}$$

Ex. 2:

$$f(z) = \cot z \quad , \quad R(0) = z\frac{\cos z}{\sin z}\Big|_{z \to 0} = 1 \tag{13}$$

- Quotient rule: If $f(z) = g(z)/h(z)$, with $g(z)$ and $h(z)$ analytic, $g(z_0) \neq 0$ and $h(z_0) = 0$ (isolated sing. pt.), as well as $h'(z_0) \neq 0$. Then

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \tag{14}$$

Ex. :

$$f(z) = \frac{\sin z}{1 - z^4} \tag{15}$$

$$R(i) = \frac{\sin z}{-4z^3}\Big|_{z=i} = \frac{\sin i}{4i} = \frac{1}{4}\sinh 1 \tag{16}$$

- If z_0 is pole of $f(z)$ of order n,

$$R(z_0) = \frac{1}{(n-1)!}\left\{\left(\frac{d}{dz}\right)^{n-1}[(z-z_0)^n f(z)]\right\}_{z \to z_0} \tag{17}$$

Ex. :

$$f(z) = \frac{z \sin z}{(z-\pi)^3} \tag{18}$$

$$R(\pi) = \frac{1}{2!}\frac{d^2}{dz^2}(z \sin z)|_{z \to \pi} = -1 \tag{19}$$

- If z_0 is an essential singularity, there is no general trick to find the residue other than constructing the LS expansion explicitly.

13.3 Definite integrals

This is simply a series of examples meant to illustrate the use of the rules listed above. If you do enough of these, including exercises in the text, you'll start to get a feel of how to use the residue theorem to evaluate various kinds of definite integrals.

1.

$$I = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} \tag{20}$$

Let's use the same trick here we used in the pf. of Cauchy's integral: take z to lie on a unit circle, such that $z = e^{i\theta}$, $dz = id\theta e^{i\theta}$. Therefore

$$d\theta = \frac{dz}{iz} \quad ; \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}. \tag{21}$$

Since θ runs from $0 \to \pi$, the integral $\int_0^{2\pi}$ we have is equivalent to z running around the unit circle contour in the complex plane, see figure. Now our

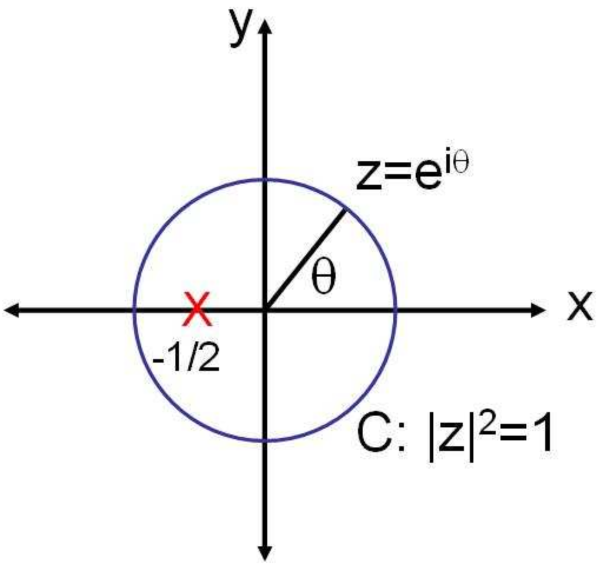

Figure 3: z runs in contour around unit circle, enclosing pole at -1/2.

integral can be represented as

$$\begin{aligned} I &= \int_C \frac{dz}{iz} \frac{1}{5 + 4\left(\frac{z+\frac{1}{z}}{2}\right)} = \frac{1}{i}\int_C \frac{dz}{5z + 2z^2 + 2} \\ &= \frac{1}{i}\int_C \frac{dz}{(2z+1)(z+2)} = \frac{1}{2i}\int_C \frac{dz}{(z+\frac{1}{2})(z+2)}. \end{aligned} \tag{22}$$

Note in the last step I factored out a factor of 2 in order to put the integral in "standard form", displaying explicitly that the integrand has a pole at $z = -1/2$ and one at $z = -2$. Only the $-1/2$ pole is inside the contour we chose, however, so we may apply the residue theorem to this pole directly and write $I = \frac{1}{2i} \cdot 2\pi i \cdot R(-1/2)$, where

$$\begin{aligned} R(-1/2) &= \lim_{z\to -1/2} (z + \frac{1}{2}) \frac{1}{(z+\frac{1}{2})(z+2)} = \frac{2}{3}, \quad \text{so} \\ I &= \frac{2\pi}{3} \end{aligned} \tag{23}$$

Note if we had wanted to find

$$\int_0^{\pi} \frac{d\theta}{4 + 5\cos\theta} \tag{24}$$

instead of $\int_0^{2\pi}$, you might thing we couldn't use the same trick, since we couldn't close the circle. The problem just requires a little preliminary reformulation, however. Since $\cos\theta$ is even in θ, the integral of our function $\int_0^{\pi}$ is equal to $\int_{\pi}^{2\pi}$, so we may write

$$\int_0^{\pi} \frac{d\theta}{4 + 5\cos\theta} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{4 + 5\cos\theta}, \tag{25}$$

and indeed use the same trick after all.

2.

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \tan^{-1} x|_{-\infty}^{\infty} = \pi, \tag{26}$$

which is obviously doable by elementary means. However we could do it with a contour integral as well, First define an integral which looks suspiciously similar, but is in the complex plane rather than on the real axis,

$$I' = \int_C \frac{dz}{1 + z^2} = 2\pi i R(i) = 2\pi i \cdot \frac{1}{2i} = \pi, \tag{27}$$

where the contour C runs on the real axis from $-\rho$ to ρ, and then gets continued to close the contour in an arc in the upper half plane as shown. The result, obtained above from the residue of the pole at i (the one enclosed by the contour), is the same as I. The idea now is to show that the contribution from the arc vanishes as $\rho \to \infty$, whereas the contribution on the real axis will be just the I we want (so if we hadn't been able to do I by elementary means we could have used the residue theorem).

The first conclusion–arc integral vanishing as $\rho \to 0$ follows from putting the integrand in polar coordinates along the arc, where $z = \rho e^{i\theta}$ again. Then the part along the arc may be written

$$\int_{\text{arc}} \frac{dz}{1+z^2} = \int_0^{\pi} \frac{d\theta \rho e^{i\theta}}{1+\rho^2 e^{2i\theta}} \underset{\rho\to\infty}{\longrightarrow} 0, \tag{28}$$

in other words the integrand gets very small as $\rho \to \infty$, *and* the measure of integration doesn't change (just the polar angle integration, which is independent of ρ.

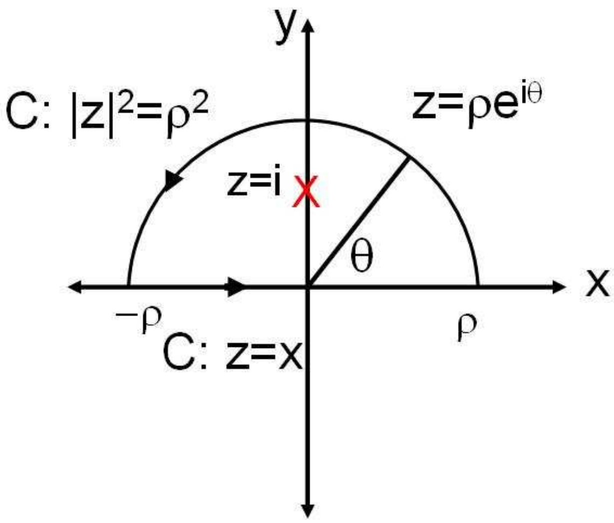

Figure 4: z runs in contour C around semicircle of radius ρ in upper half-plane, and then along real axis, enclosing pole at $z = i$.

The next point is simply that the part of C along the real axis, once we let $\rho \to \infty$, will just be the integral we want, since along this path $z = x$. So we have shown that

$$I = I' = \pi, \tag{29}$$

as we found by elementary means. On the HW you will have to do several examples which are not so easy by elementary means, but can be done quickly by complex contour integration.

3.

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx \tag{30}$$

Now this is a tricky integral. If you plot the function, it oscillates rapidly but has a limiting value of 1 as $x \to 0$. What we want to do, as usual, is to find an analogous integral in the complex plane, where the integrand reduces to our integrand on the real axis. Unlike the last time, however, if we choose the obvious choice

$$I' = \oint_C \frac{e^{iz}}{z} dz, \tag{31}$$

with contour C left unspecified for the moment, we have a problem, because the numerator e^{iz} is analytic everywhere, whereas $1/z$ has a pole at $z = 0$. Where do we run our contour? If we run it straight through the singularity, we are not really allowed to use Cauchy's theorem. So we have to take a little detour in a carefully controlled way. The figure shown shows one such solution: we will compute

$$I' = \oint_{C=C_1+C_2+C_3+C_4} \frac{e^{iz}}{z} dz, \tag{32}$$

where the segments of the contour C_i are shown in the figure. As someone pointed out in class, we could have taken a detour in the lower half plane as well. However this would then have the effect of enclosing the singularity at zero, so we would have to calculate its residue. The contour shown has the benefit that the integral around it is manifestly zero, since there are no singularities inside!

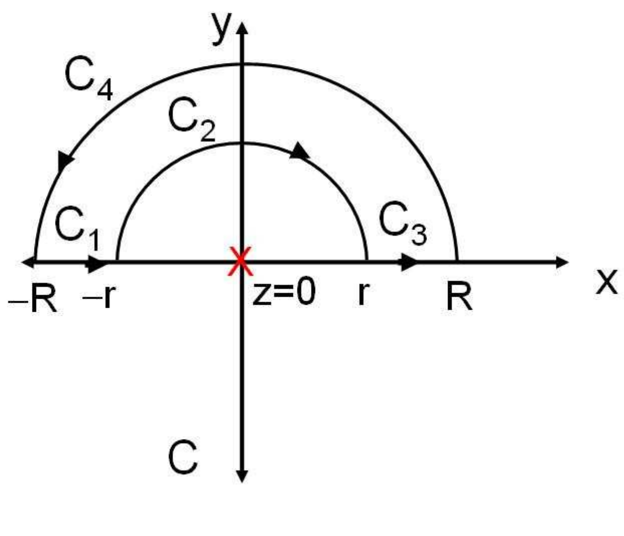

Figure 5: z runs in contour C around semicircle of radius ρ in upper half-plane, and then along real axis, enclosing pole at $z = i$.

How then does it help us? Let's consider the contributions from the various paths, keeping in mind that we want to take the limit eventually $R \to \infty$ and $r \to 0$. Then the arc at infinity C_4 will contribute nothing as before (think a little about why–it's a bit subtle, but true). The two contours on the real axis, C_1 and C_3, will be extended to as to occupy the entire real axis, and the integral will in fact reduce to exactly what we want. Finally, there's the integral around the singularity C_2. This will give us some answer. So the strategy is to use the fact that

$$I' \equiv \oint_{C=C_1+C_2+C_3+C_4} \frac{e^{iz}}{z} dz = 0 = \int_{C_1+C_3} \frac{e^{ix}}{x} dx + \lim_{r\to 0} \int_{C_2} \frac{e^{iz}}{z} dz. \quad (33)$$

But the C_2 part is quite similar to those circular path integrals we did before. Write $z = re^{i\theta}$, $dz = rid\theta e^{i\theta}$, so

$$\int_{C_2} \frac{e^{iz}}{z} dz = \int_{C_2} e^{iz} i d\theta \quad (34)$$

Now when you take the limit $r \to 0$, e^{iz} is completely regular at $z \to 0$, so we may replace it by its limiting value 1. So

$$\int_{C_2} \frac{e^{iz}}{z} dz = \int_{\pi}^{0} i d\theta = -i\pi. \quad (35)$$

So finally the equation for the contour integral reads

$$I' = \int_{-\infty}^{\infty} \frac{e^{ix}}{x} - i\pi = 0, \quad (36)$$

or taking real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x} = 0 \quad ; \quad \int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi \quad (37)$$

This answers our question, but it raises another. It is perhaps no surprise that $\int_{-\infty}^{\infty} \frac{\cos x}{x} = 0$, since the integrand is odd in x. However the integrand diverges at $x = 0$ also; the singularities to the left and right of 0 cancel each other. Since the singularities are individually not integrable (if we were to integrate $\int_0^{\infty}$ for example, we would get ∞) we know the value we get is sensitive to exactly how we take the limit. If one takes it symmetrically (see text), the result is well-defined even thought the integral is improper, and is called the *Cauchy principal value* of the integral. See Boas for more discussion and another example.