14 Fourier analysis

Read: Boas Ch. 7.

14.1 Function spaces

A function can be thought of as an element of a kind of vector space. After all, a function $f(x)$ is merely a set of numbers, one for each point x of the underlying space. We can add functions in this way, componentwise, like vectors $h(x) = f(x)+g(x)$, and (we will show below), we can define a metric, or distance function, on the set of all functions as well. It's simplest to think about 1D first, a finite interval $0 \le x \le 2L$, and imagine "discretizing" this space so the N points in it are separated, like the gradations on a ruler by an amount $\Delta \equiv 2L/N$. A "vector" in function space $|f\rangle$ is therefore defined to be the set of components $f_1, f_2, \ldots f_N$ representing the values of the function f at the points x_1, x_2, ... Now if we choose a basis of this space called a "position basis", we define a vector

$$|i\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{1}$$

where the 1 is in the ith position, in other words the vector represents the position x_i. This is clearly a basis for the vector space, since each vector is linearly independent and the whole space is spanned. The function f may now be represented as

$$|f\rangle = f_1|0\rangle + f_2|1\rangle + f_3|4\rangle \cdots + f_N|N\rangle, \tag{2}$$

i.e. the function has the value $f_1 \equiv f(x_1)$ at x_1, and so on. Note this space is finite-dimensional, but we can make L as large as we like, or choose N as large as we like..

Now suppose I wanted to define the product of two functions on this space. Well, a function has been represented as a vector, so the idea is obvious: define a scalar product as you would between two vectors:

$$\langle f|g\rangle = \sum_{i=1}^{N} f_i^* g_i \tag{3}$$

Now however the idea is to take the limit $N \to \infty$ so that we have an infinite dimensional vector space (functions still 'live" on a finite interval however!). We can do this in such a way such that the limit is well behaved if we define

$$\langle f|g\rangle = \sum_{i=1}^{N} f_i^* g_i \Delta \to \int_0^{2L} f^*(x)g(x)dx. \tag{4}$$

Once we have an inner product, we can define the lengths of vectors (functions), i.e.

$$|f| \equiv \sqrt{\langle f|f\rangle} = \int_0^{2L} |f(x)|^2 dx. \tag{5}$$

A function is said to be *normalized* if its length is one, i.e.

$$|f| = \int_0^{2\pi} |f(x)|^2 = 1. \tag{6}$$

A space of functions where all elements are normalized is called a *Hilbert space*, after mathematician David Hilbert.

We can also define the notion of orthogonality, i.e. two functions are *orthogonal* over the interval $[0, 2L]$ if

$$\langle f|g\rangle = \int_0^{2L} f^*(x)g(x) = 0 \qquad \Rightarrow \quad f, g \text{ orthogonal} \tag{7}$$

.

There can be many different sets of orthogonal functions. Here are some examples:

1. $e^{i\pi mx/L}$ over interval $(0, 2L)$. Define

$$|m\rangle = \frac{1}{\sqrt{2L}} e^{i\pi mx/L}, \; m{=}0, \pm 1, \pm 2 \tag{8}$$

   You can check that

$$\langle m||n\rangle = \frac{1}{2L} \int_0^{2L} e^{-i\pi mx/L} e^{i\pi nx/L} dx = \delta_{mn} \tag{9}$$

2. *Legendre polynomials* on interval $(-1, 1)$.

$$\int_{-1}^{1} dx P_n(x) P_m(x) = \frac{2}{2n+1} \, \delta_{mn}. \tag{10}$$

Note as defined these functions are orthogonal set, but not normalized, since their square integral isn't 1. (These are solutions to a particular differential equation which arises in electromagnetic theory and quantum mechanics.) Without further discussion about where the P_n come from, I can give you the first few so that you can test Eq. (10):

$$P_0(x) = 1 \ ; \quad P_1(x) = x \ ; \quad P_2(x) = (3x^2 - 1)/2 \ ; \quad P_3(x) = (5x^3 - 3x)/2 \tag{11}$$

14.2 Fourier series

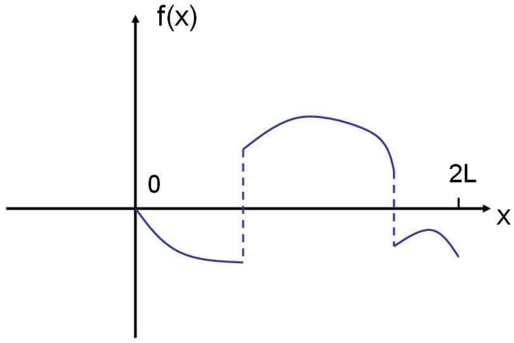

Figure 1: Example of function satisfying Dirichlet conditions.

Let's focus on the $|m\rangle$ set a little more closely. This set of functions form an orthonormal basis for functions obeying "Dirichlet conditions" (Fig. 1).

- periodic (period $2L$)
- single-valued on interval
- finite number of max/min
- finite number of discontinuities
- $\int_0^{2L} |f(x)|^2 dx$ finite.

In other words, any function $f(x)$ obeying these conditions can be expanded in this basis,

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{i\pi m x/L} \tag{12}$$

(compare

$$|f\rangle = \sum_1^N c_m |m\rangle. \tag{13}$$

How do we figure out what the c_m are? For ordinary vectors, we can just use the orthonormality of the basis: take the inner product of (13) with $\langle n|$ to find

$$\langle n|f\rangle = \sum_{m=1}^{N} c_m \underbrace{\langle n|m\rangle}_{\delta_{mn}.} = c_n \tag{14}$$

$$\tag{15}$$

We can use the same trick with our complex exponential basis: the coefficient c_m in (12) is

$$c_m = \langle m|f\rangle = \frac{1}{2L}\int_0^{2L} e^{-i\pi mx/L} f(x). \tag{16}$$

These are called *Fourier coefficients.*

Note: if $L = \pi$, we have an expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, c_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)dx \tag{17}$$

$$\tag{18}$$

period 2π. This is called an exponential Fourier series, or just Fourier series.

14.3 Sine and cos Series

Recall $e^{inx} = \cos nx + i \sin nx$, so

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} c_n(\cos nx + i\sin nx) &= c_0 + (\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty}) c_n(\cos nx + i\sin nx) \\
&= c_0 + \sum_{n=1}^{\infty}[(c_n + c_{-n})\cos nx + i(c_n - c_{-n})\sin nx] \\
&\equiv \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),
\end{aligned} \tag{19}$$

where

$$a_n = c_n + c_{-n} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \quad ; \quad b_n = i(c_n - c_{-n}) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx.$$

Ex. 1

Piecewise continuous function. Suppose you want to Fourier analyze the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases} \tag{20}$$

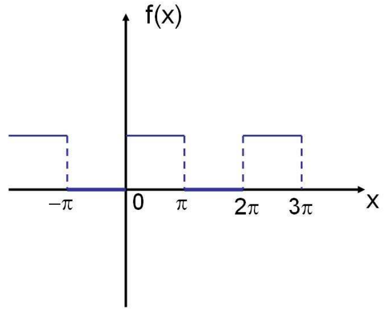

Figure 2:

First extend it periodically, as in the figure. Then the Dirichlet conditions are fulfilled, and we can immediately write

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \;=\; \frac{1}{\pi}\int_0^{\pi} dx = 1 \;\; ; \quad a_n = \frac{1}{\pi}\int_0^{\pi} \cos nx dx = 0 \;\; , n = 1, 2, \ldots \tag{21}$$

$$b_n \;=\; \frac{1}{\pi}\int_0^{\pi} \sin nx dx = \begin{cases} 2/(n\pi) & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{22}$$

So Fourier series for $f(x)$ is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\right). \tag{23}$$

Now it should seem slightly crazy to you that we can add a bunch of sine functions and get something flat and piecewise continuous. But it works! How it works, i.e. how the series converges to the "right answer" in this case, is shown in the next figure.

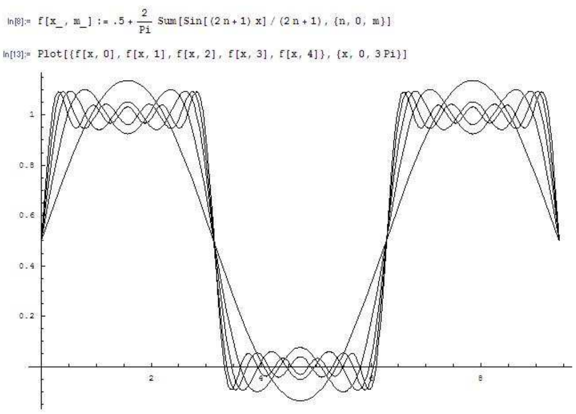

Figure 3: How one builds up a square wave function as a sum of sine waves.

<u>Ex. 2</u>

$$f(x) = x^2 \quad -\pi < x < \pi. \tag{24}$$

Note that $f(x)$ is even in x. We therefore know in advance that the series for f is a cosine series only ($b_n = 0$):

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3} \tag{25}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi}(-1)^n \frac{2\pi}{n^2} = (-1)^n \frac{4}{n^2}. \tag{26}$$

So we get a Fourier series expansion for x^2:

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}. \tag{27}$$

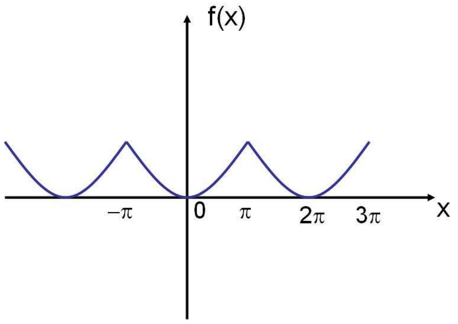

Figure 4:

<u>Ex. 3</u>

$$f_1(x) = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases} \tag{28}$$

$$f_2(x) = \begin{cases} x & 0 < x < \pi \\ x & -\pi < x < 0 \end{cases} \tag{29}$$

The function f_1 is even in x, so we expect *a priori* to find it represented by a cosine series, i.e. all b_n=0. On the other hand f_2 is odd, so it will be represented by a sine series. Note that over the interval $0 < x < \pi$, they represent the same function!

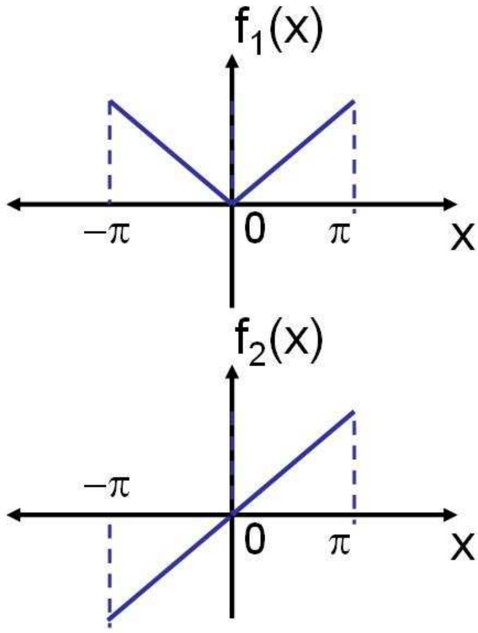

Figure 5: Ex. 3 $f_1(x)$ and $f_2(x)$.

14.4 Fourier integral

Now here we have just analyzed a function over a symmetric interval $[-\pi, \pi]$, and we can clearly do the same over a symmetric interval $[-L, L]$.