

# 4 Multiple integrals; vectors

## 4.1 Multiple integrals

Let's review this subject by doing various examples of integrating a function  $f(x, y)$  over a region of 2-space:

Ex. 1:

$$I = \int_{\text{region}} y\sqrt{x} dy dx = \int_0^4 dx \sqrt{x} \int_0^{\sqrt{x}} y dy = \int_0^4 dx \sqrt{x} \cdot \frac{1}{2}x = \frac{32}{5} \quad (1)$$

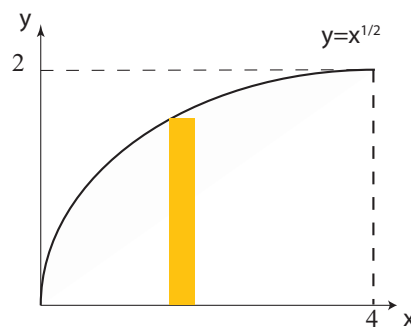


Figure 1: Ex. 1

Q: how about changing the order of integration? We could do the  $x$ -integral first obviously. But we need to be careful of the order of the limits.:

$$I = \int_{y=0}^2 dy y \int_{x=y^2}^4 \sqrt{x} dx = \frac{32}{5}. \quad (2)$$

Q: Why change the order? Sometimes it's important:

Ex. 2:

$$I = \int_{x=0}^{\ln 16} dx \int_{y=e^{x/2}}^4 \frac{dy}{\ln y} \quad (3)$$

Can't do  $\int dy/\ln y$ , so switch:

(4)

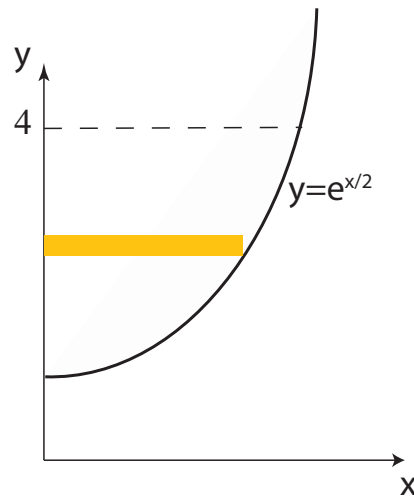


Figure 2: Ex. 1

So let's draw a new figure (always draw a figure if you switch!).

$$I = \int_{y=1}^4 \frac{dy}{\ln y} \int_0^{2 \ln y} dx = \int_{y=1}^4 dy \cdot 2 = 6 \quad (5)$$

## 4.2 Change of variables: the Jacobian

First, let's do a standard example where we don't get into formalities:

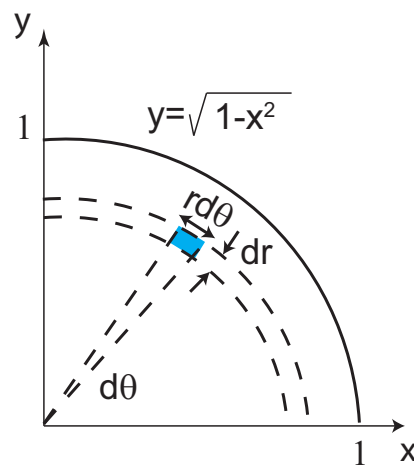


Figure 3: Ex. 1

Ex. 3:

$$I = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)} dy dx \quad (6)$$

This will certainly be easier in polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^1 e^{-r^2} r dr d\theta = \frac{\pi}{4}(1 - 1/e). \quad (7)$$

Note the measure for the integral in polar coords.,

$$dx dy = r dr d\theta, \quad (8)$$

is just the size of the little area element in Fig. 3, but can be obtained formally. In general the change of variables in 2D is defined

$$\int \int_R f(u, v) du dv = \int \int_{R'} f(u(r, s), v(r, s)) \cdot \left| J \left( \frac{u, v}{r, s} \right) \right| dr ds, \quad (9)$$

where  $R$  is a region in 2D and  $R'$  is the transformed region, which need not have the same shape.  $J$  is the ‘‘Jacobian of transformation’’

$$J \left( \frac{u, v}{r, s} \right) = \left| \begin{array}{cc} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{array} \right| = \frac{1}{J \left( \frac{r, s}{u, v} \right)}, \quad (10)$$

i.e. the determinant of the partial derivatives as shown. If you have forgotten how to take a determinant, go ahead and look it up in Boas p. 89. Of course the concept can be generalized to higher-D transformations  $(x, y, z, \dots) \rightarrow (u, v, w, \dots)$ , but the determinantal form remains, one just has to do a bit more work. Let’s specialize now to polar coordinates. Take  $u = x = r \cos \theta$ ,  $v = y = r \sin \theta$ , so

$$J \left( \frac{x, y}{r, \theta} \right) = \left| \begin{array}{cc} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{array} \right| = r(\cos^2 \theta + \sin^2 \theta) = r, \quad (11)$$

so  $dx dy = r dr d\theta$ , what we worked out already in more intuitive fashion.

Here’s a slightly tricky example where the shape of the transformed region is quite different.

Ex. 5:

$$I = \int_0^1 dx \int_0^x \frac{(x+y)e^{x+y}}{x^2} dy \quad (12)$$

Let’s make a transformation  $x, y \rightarrow u, v$  where  $u = y/x$ ,  $v = x + y$ . The choice of  $v$  is somewhat straightforward; why  $u$  is chosen I have no idea. Anyway, 1st task

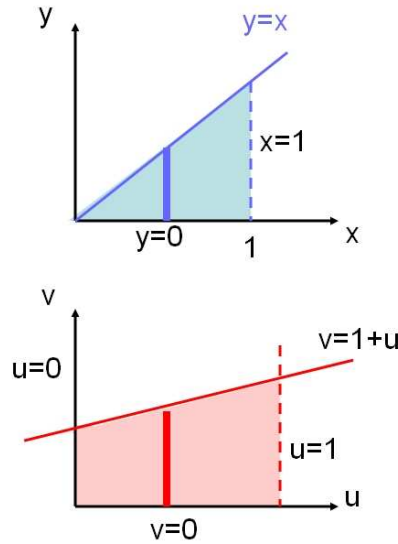


Figure 4: Ex. 5. Original and transformed regions.

is to calculate Jacobian:

$$J \left( \frac{u, v}{x, y} \right) = \begin{vmatrix} -y/x^2 & 1/x \\ 1 & 1 \end{vmatrix} = -\frac{x+y}{x^2} \quad (13)$$

$$\Rightarrow \left| J \left( \frac{x, y}{u, v} \right) \right| = \frac{x^2}{x+y}. \quad (14)$$

So

$$I = \int \int \frac{ve^v}{x^2(v)} \frac{x^2(v)}{v} dudv = \int \int e^v dudv, \quad (15)$$

nice and simple, but what are the transformed limits? Well, starting from the original ones, we learn

$$y = 0 \Rightarrow u = 0 \text{ if } x \neq 0 \quad ; \quad y = x \Rightarrow u = 1 \quad (16)$$

$$x = 1 \Rightarrow v = 1 + u \quad ; \quad y = x = 0 \Rightarrow v = 0. \quad (17)$$

So

$$I = \int_0^1 du \int_0^{1+u} dv e^v = \int_0^1 du (e^{1+u} - 1) = e^2 - e - 1. \quad (18)$$

**measures for cylindrical and spherical coords.:** (Derived in Boas)–check!

Spherical  $x = r \sin \theta \cos \phi$  ;  $y = r \sin \theta \sin \phi$  ;  $z = r \cos \theta$ :

$$\left| J \left( \frac{x, y, z}{r, \theta, \phi} \right) \right| = r^2 \sin \theta \quad (19)$$

$$\text{i.e. } dx dy dz \rightarrow r^2 \sin \theta dr d\theta d\phi \quad (20)$$

Cylindrical  $x = \rho \cos \theta$  ;  $y = \rho \sin \theta$ ;  $z = z$  :

$$\left| J \left( \frac{x, y, z}{\rho, \theta, z} \right) \right| = \rho \quad (21)$$

$$\text{i.e. } dx dy dz \rightarrow \rho d\rho d\theta dz \quad (22)$$

**(N.B.  $\theta$  in spherical coords. is not the same  $\theta$  as for cylindrical coords! Might want to use  $\phi$  for cylindrical coords instead.)**

Ex. 6:

Calculate the moment of inertia of a cone of height equal to its base radius  $h = R$ . Take the density of the material to be  $\rho_0$ , assumed homogeneous. Moment of inertia is then

$$I = \int \int \int_V \rho_0 dV (x^2 + y^2) = \rho_0 \int_{z=0}^h dz \int_{r=0}^z r dr \int_0^{2\pi} d\theta \quad r^2 = \rho_0 \frac{\pi h^5}{10}. \quad (23)$$

Note dimensions are correct, since  $[\rho_0] = M/L^3$ , so  $[\rho_0 h^5] = ML^2$ .

## 4.3 Vectors

### 4.3.1 Properties of vectors

Vector: “set of components which transforms under rotation of a coordinate system in the same way as the coordinates of a point  $\vec{r}$ . ” (“A vector is something that transforms like a vector”.) Huh? What does that mean? Take the coordinates  $x_1, x_2$  of a point  $\vec{r}$  in a Cartesian coordinate system.

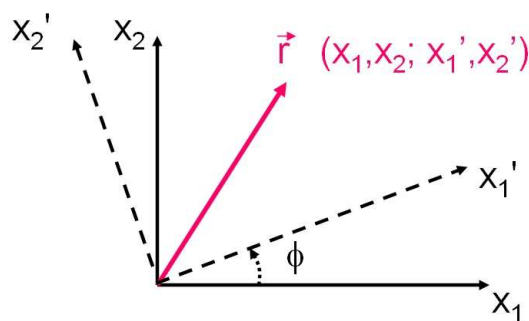


Figure 5: Transformation of coordinates  $x_1, x_2 \rightarrow x_1', x_2'$ .

If we rotate the coordinate axes to a new set  $x_1', x_2'$  by an angle  $\phi$ , geometry tells

us

$$x'_1 = \underbrace{\cos \phi}_{a_{11}} x_1 + \underbrace{\sin \phi}_{a_{12}} x_2 \quad (24)$$

$$(25)$$

$$x'_2 = \underbrace{-\sin \phi}_{a_{21}} x_1 + \underbrace{\cos \phi}_{a_{22}} x_2 \quad (26)$$

$$(27)$$

so we can write this as a general linear transformation

$$x'_i = a_{i1}x_1 + a_{i2}x_2 = \sum_{j=1}^2 a_{ij}x_j. \quad (28)$$

What happens to the vector  $\vec{r}$  under this transformation?

$$\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 \quad (29)$$

$$= x'_1\hat{e}'_1 + x'_2\hat{e}'_2 \quad (30)$$

$$(31)$$

$$x'^2_1 + x'^2_2 = x^2_1 + x^2_2 \quad (32)$$

$$(33)$$

$$\Rightarrow a^2_{11} + a^2_{21} = 1 \quad ; \quad a^2_{12} + a^2_{22} = 1 \quad (34)$$

$$a_{11}a_{12} + a_{21}a_{22} = 0. \quad (35)$$

The length of the vector is preserved in the new coordinate system; the transformation is said to be *orthogonal*. We can write the same thing in a nice compact notation using the "Kronecker delta" symbol

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (36)$$

as

$$\sum_i a_{ij}a_{ik} = \delta_{jk} = \sum_i a_{ji}a_{ki}. \quad (37)$$

If you find this confusing, write this out in terms of indices and see what it means. You may also wish to ask yourself how it's related to matrix multiplication.

Inverse transformation: ( $\phi \rightarrow -\phi$ ):

$$a_{ij} = \frac{\partial x'_i}{\partial x_j} = \frac{\partial x_j}{\partial x'_i} \quad (38)$$

(Why is this true?)

Some properties of general N-D vectors  $\vec{A} = (A_1, A_2 \dots)$ :

- $\vec{A} = \vec{B} \Rightarrow A_i = B_i$
- $\vec{A} + \vec{B} = \vec{C} \Rightarrow A_i + B_i = C_i$
- a  $\vec{A} = (aA_1, aA_2, \dots, aA_N)$
- $-\vec{A} = (-A_1, -A_2, \dots, -aA_N)$
- $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
- $a(\vec{A} + \vec{B}) = a\vec{A} + b\vec{B}$

etc.

Magnitude of a vector:

$$A^2 = \sum_{i=1}^N A_i^2 = \sum_{i=1}^N A_i'^2 \quad \text{check!} \quad (39)$$

$$|\vec{A}| = \sqrt{A^2} = A \quad (40)$$

#### 4.3.2 Products of vectors

1) Dot or scalar product:

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i \quad (41)$$

2) Cross product

$$(\vec{A} \times \vec{B})_i = \sum_{jk} \epsilon_{ijk} A_j B_k, \quad (42)$$

where  $\epsilon_{ijk}$  is the so-called Levi-Civita symbol, sometimes called the completely antisymmetric tensor:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two indices are the same} \\ +1 & \text{if indices correspond to an **even** permutation of 123} \\ -1 & \text{if indices correspond to an **odd** permutation of 123} \end{cases} \quad (43)$$

Very useful identity – worth memorizing!

$$\sum_k \epsilon_{ijk} \epsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \quad (44)$$

Ex 1:

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = ? \quad (45)$$

$$\begin{aligned} \sum_i (\vec{A} \times \vec{B})_i (\vec{C} \times \vec{D})_i &= \sum_i \sum_{jk} \epsilon_{ijk} A_j B_k \cdot \sum_{\ell m} \epsilon_{i\ell m} C_\ell D_m \\ &= \sum_{ijklm} (\epsilon_{ijk} \epsilon_{i\ell m}) A_j B_k C_\ell D_m \\ &= \sum_{jklm} (\delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{kl}) A_j B_k C_\ell D_m \\ &= \sum_{\ell m} (A_\ell B_m C_\ell D_m - A_m B_\ell C_\ell D_m) \\ &= \left( \sum_\ell A_\ell C_\ell \right) \left( \sum_m B_m D_m \right) - \left( \sum_\ell B_\ell C_\ell \right) \left( \sum_m A_m D_m \right) \\ &= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{B} \cdot \vec{C})(\vec{A} \cdot \vec{D}) \end{aligned} \quad (46)$$

Some more identities to check using these techniques:

- $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$
- $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$  (“BAC-CAB rule”)

Remarks:

- 2 or 3d  $\vec{A} \cdot \vec{B} = AB \cos \theta_{AB}$ . If  $\vec{A} \cdot \vec{B} = 0$ , two vectors are “orthogonal”,  $\theta_{AB} = \pi/2$ .
- unit vectors  $\hat{e}_1, \hat{e}_2, \dots$ . We can choose mutually orthogonal,  $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ . Also note  $\hat{e}_i \times \hat{e}_j = \epsilon_{ijk} \hat{e}_k$ .
- Expand any vector  $\vec{A}$  in terms of  $D$  orthogonal unit vectors in  $D$  dimensions,  $\vec{A} = \sum_i A_i \hat{e}_i$ .
- The scalar or dot product of two vectors is *invariant* under coordinate transformations:

$$\begin{aligned} \vec{A}' \cdot \vec{B}' &= \sum_i A'_i B'_i = \sum_i \left( \sum_j a_{ij} A_j \right) \left( \sum_k a_{ik} B_k \right) = \sum_{ijk} a_{ij} a_{ik} A_j B_k \\ &= \sum_{jk} \delta_{jk} A_j B_k = \sum_k A_k B_k = \vec{A} \cdot \vec{B} \end{aligned} \quad (47)$$

- $|\vec{A} \times \vec{B}| = AB \sin \theta_{AB}$



- meaning of  $|\vec{A} \cdot (\vec{B} \times \vec{C})|$ : volume of parallelepiped spanned by vectors  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$ .