

# 6 Div, grad curl and all that

## 6.1 Fundamental theorems for gradient, divergence, and curl

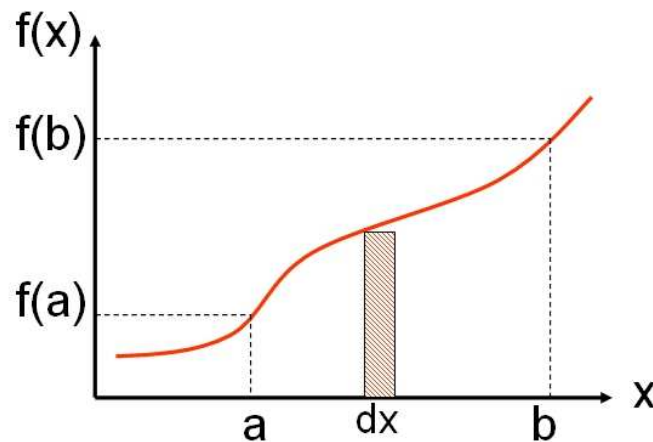


Figure 1: Fundamental theorem of calculus relates  $df/dx$  over  $[a, b]$  and  $f(a)$ ,  $f(b)$ .

You will recall the fundamental theorem of calculus says

$$\int_a^b \frac{df(x)}{dx} dx = f(b) - f(a), \quad (1)$$

in other words it's a connection between the rate of change of the function over the interval  $[a, b]$  and the values of the function at the endpoints (boundaries) of that interval. There are equivalent “fundamental theorems” for line integrals, area integrals, and volume integrals. In vector calculus we deal with different types of changes of scalar and vector fields, e.g.  $\vec{\nabla}\phi$ ,  $\vec{\nabla} \cdot \vec{v}$ , and  $\vec{\nabla} \times \vec{v}$ , & each has its own theorem.

We've discussed line integrals before, mostly in the context of the work done along a path, but let's remind ourselves of the definition:

$$\int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \vec{F}(\vec{r}_i) \cdot d\vec{r}_i, \quad (2)$$

in other words we add up the area of all the little “rectangles”  $\vec{F}(\vec{r}_i) \cdot d\vec{r}_i$  consisting of the vector  $\vec{F}$  at the point  $\vec{r}_i$  dotted into the path element  $d\vec{r}_i$ , see Figure. Remember the point is that although we are doing an integral in a 2D space, we are constrained to move along a path, so there is only one real independent variable.

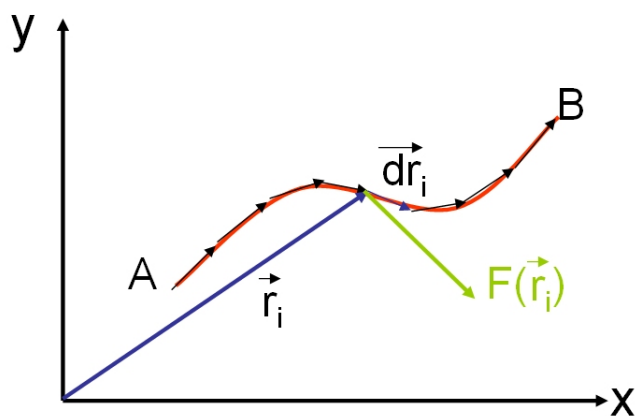


Figure 2: Line integral.

Example from test: Consider the triangle in the  $(x, y)$  plane with vertices at  $(-1, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ . Evaluate the closed line integral

$$I = \oint (-y\hat{x} + x\hat{y}) \cdot d\vec{r} \quad (3)$$

around the boundary of the triangle in the anticlockwise direction.

Since the vector  $\vec{r}$  has components  $(x, y)$ , the measure is  $d\vec{r} = \hat{x}dx + \hat{y}dy$ , so  $(-y\hat{x} + x\hat{y}) \cdot d\vec{r} = -ydx + xdy$ .

On leg  $(-1, 0) \rightarrow (1, 0)$  we have  $y = 0$ , so integral is  $\int_{-1}^1 (-y)dx = 0$ . On the leg  $(1, 0) \rightarrow (0, 1)$  we have  $y = -x + 1$ , so integral is  $-\int_1^0 (-x + 1)dx + \int_0^1 (1 - y)dy = \frac{1}{2} + \frac{1}{2} = 1$ . On the path  $(0, 1) \rightarrow (-1, 0)$  we have  $y = x + 1$ , so integral is  $-\int_0^{-1} (x + 1)dx + \int_1^0 (y - 1)dy = \frac{1}{2} + \frac{1}{2} = 1$ . So total line integral is 2.

Let's go back and look at the leg from  $(1, 0) \rightarrow (0, 1)$  again. We could have "parameterized" this leg as  $x = 1 - t$ ,  $y = t$ ,  $0 < t < 1$ . Then  $\vec{r} = (1 - t)\hat{x} + t\hat{y}$ , and  $d\vec{r} = (-\hat{x} + \hat{y})dt$ . We can therefore write this integral as

$$\int_{(1,0)}^{(0,1)} \vec{F} \cdot d\vec{r} = \int_0^1 (-t\hat{x} + (1 - t)\hat{y}) \cdot (-\hat{x} + \hat{y})dt = \int_0^1 (t + 1 - t) = 1, \quad (4)$$

same answer as above.

### 6.1.1 Conservative field.

Recall we said that the integral of an exact differential was independent of the path. This result can be expressed in our current particular context, line integrals of vector fields. Suppose you have a vector field  $\vec{F}$  which can be expressed as the

gradient of a scalar field,  $\vec{F} = \vec{\nabla}\phi$ . Then we have the fundamental theorem for gradients:

$$\int_a^b \vec{\nabla}\phi \cdot d\vec{r} = \int_a^b d\phi = \phi(b) - \phi(a), \quad (5)$$

in other words the integral  $\int_a^b \vec{F} \cdot d\vec{r}$  doesn't depend on the path between  $a$  and  $b$ . If  $\vec{F} = \vec{\nabla}\phi$  for some  $\phi$ ,  $\vec{F}$  is *conservative*, integral independent of path. Since as you proved on the practice test,  $\vec{\nabla} \times \vec{\nabla}\phi = 0$  for any  $\phi$ , showing that  $\vec{\nabla} \times \vec{F} = 0$  is a sufficient condition for a force to be conservative as well.

### 6.1.2 Surface integrals

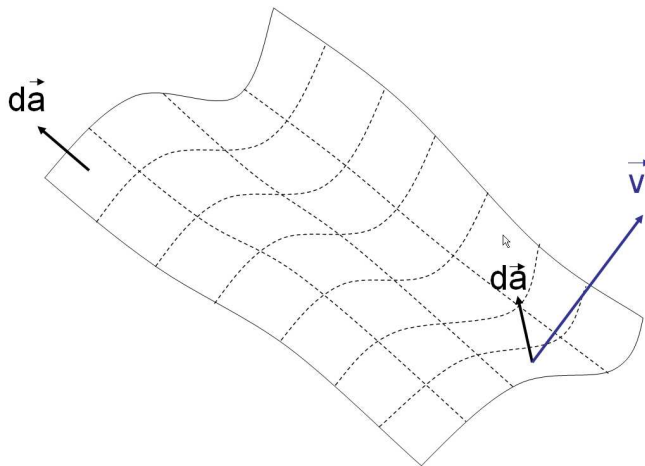


Figure 3: Arbitrary surface with measure  $d\vec{a}$ .

We define the surface integral of the vector field  $\vec{v}$  as

$$\int_A \vec{v} \cdot d\vec{a} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \vec{v}(\vec{r}_i) \cdot d\vec{a}_i, \quad (6)$$

where the infinitesimal directed area element  $d\vec{a}_i$  is oriented perpendicular to the surface locally. Sometimes we call the integrand the “flux” of  $\vec{v}$  through  $A$ ,  $d\Phi = \vec{v} \cdot d\vec{a}$ . This should remind you of the example of the flux of the electric field through a surface from your E&M class, but it can be defined for any vector field.

Another example: suppose  $\vec{v}$  is a current density, i.e. represents the number of particles crossing a  $\perp$  plane per unit area per unit time. Then the flux  $d\Phi = \vec{v} \cdot d\vec{a}$  is the number of particles passing through  $da$  per time. In general, the surface  $A$

may not be a plane but may have some curvature, but the total flux will just be

$$\Phi = \int_A \vec{v} \cdot d\vec{a}. \quad (7)$$

Suppose  $A$  is a closed surface now, and there is a source of particles shooting out in all directions, penetrating the surface. Clearly the number of particles passing through the surface per time per area is nonzero. On the other hand, if the source is outside  $A'$ , as shown, the flux through the surface looks like it might cancel on the left and right surfaces and be zero.<sup>1</sup> In fact, we must somehow be able to express the conservation law that the particles which enter  $A'$  must eventually leave it: what goes in, must come out. The answer is a very general relation between the surface integral of a vector field and the volume integral of its divergence  $\vec{\nabla} \cdot \vec{v}$ .

### 6.1.3 Fundamental theorem for divergences: Gauss theorem.

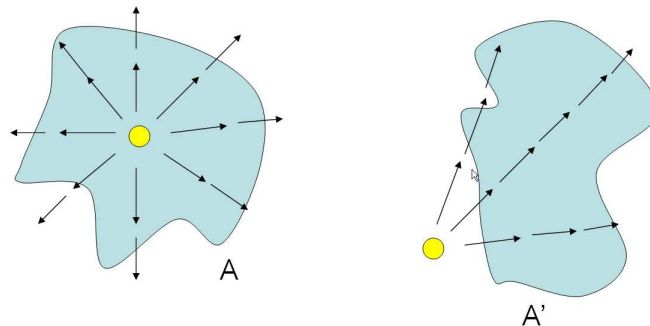


Figure 4: Left: particle source inside closed surface  $A$ . Flux is nonzero. Right: source outside closed surface. Flux through  $A'$  is zero.

Mathematically the divergence of  $\vec{v}$  is just  $\partial_i v_i = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$ . Consider the volumes inside  $A$  and  $A'$ ,  $A = \partial V$  and  $A' = \partial V'$  (the symbol “ $\partial$ ” here means “the boundary of”). Remembering that we’re thinking of  $\vec{v}$  as a current density for now, let’s ask ourselves how much how much comes out, passing through the surface. We can divide up the whole volume into infinitesimal cubes  $dx dy dz$  and consider each cube independently. First imagine that  $\vec{v}$  is constant. Then the surface integral over the cube will give zero, because the flux in one face will be exactly cancelled by the flux out the other face (signs of the dot product  $\vec{v} \cdot d\vec{a}$  will be opposite). If  $\vec{v}$  is not a constant, the vectors on opposite faces will not be quite equal, e.g. at  $x$

<sup>1</sup>You might suspect that this result is related to Gauss’s law in electromagnetism, and if the source were and electric charge, and  $\vec{v}$  the electric field, you would be right. Be careful, however. Gauss’s law is a statement about physics, and the relationship between the electric flux and its source, the electric charge. We reminded ourselves the first week that this depends sensitively on the  $1/r^2$  nature of the Coulomb force.

we'll have  $v_x$  but at  $x + dx$  we'll have  $v_x + dv_x$ . So the flow rate out of the cube in this direction will be equal to  $\frac{\partial v_x}{\partial x} dx$ . Integrating over the whole cube will give the whole flow rate out:

$$\int_{\text{cube}} \vec{\nabla} \cdot \vec{v} \, dx dy dz = \int_{\partial \text{ cube}} \vec{v} \cdot d\vec{a}. \quad (8)$$

But if we consider any arbitrary volume  $\tau$  to be a sum of such cubes, the surface integrals of the interior infinitesimal cubes will cancel, because their normals  $\hat{a}$  are oppositely directed. Therefore the total volume integral over  $\tau$  is similarly related to the total surface integral of  $\partial\tau$ :

$$\int_{\tau} \vec{\nabla} \cdot \vec{v} \, d\tau = \int_{\partial\tau} \vec{v} \cdot d\vec{a}. \quad (9)$$

This is called the *divergence theorem* or Gauss's theorem (not Gauss's law!).

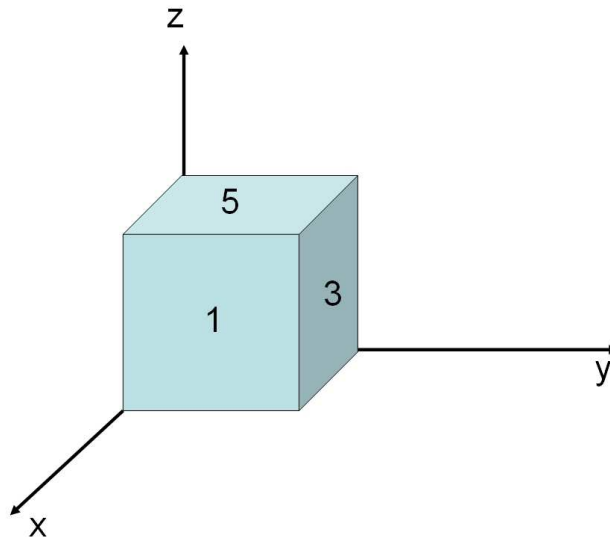


Figure 5: Faces of cube in example.

Let's do an example where the total volume  $\tau$  is an actual cube ( of length 1): take  $\vec{v} = \hat{i}y^2 + \hat{j}(2xy + z^2) + \hat{k}2yz$ . The divergence is  $\vec{\nabla} \cdot \vec{v} = 2(x + y)$ . Volume integral is

$$\int_{\tau} \vec{\nabla} \cdot \vec{v} = 2 \int_0^1 \int_0^1 \int_0^1 dx dy dz (x + y) = 2, \quad (10)$$

and the integrals through the faces are

$$\begin{aligned}
(1) \quad d\vec{a} &= dydz \hat{i}, \quad x = 1, \quad \vec{v} \cdot d\vec{a} = y^2 dydz \Rightarrow \int_0^1 \int_0^1 y^2 dydz = \frac{1}{3} \\
(2) \quad d\vec{a} &= -dydz \hat{i}, \quad x = 0, \quad \vec{v} \cdot d\vec{a} = -y^2 dydz \Rightarrow -\int_0^1 \int_0^1 y^2 dydz = -\frac{1}{3} \\
(3) \quad d\vec{a} &= dx dz \hat{j}, \quad y = 1, \quad \vec{v} \cdot d\vec{a} = (2x + z^2) dx dz \Rightarrow \int_0^1 \int_0^1 (2x + z^2) dx dz = \frac{4}{3} \\
(4) \quad d\vec{a} &= -dx dz \hat{j}, \quad y = 0, \quad \vec{v} \cdot d\vec{a} = -z^2 dx dz \Rightarrow \int_0^1 \int_0^1 -z^2 dx dz = -\frac{1}{3} \\
(5) \quad d\vec{a} &= dx dy \hat{k}, \quad z = 1, \quad \vec{v} \cdot d\vec{a} = 2yz dx dy \Rightarrow 2 \int_0^1 \int_0^1 y dx dy = 1 \\
(6) \quad d\vec{a} &= -dx dy \hat{k}, \quad z = 0, \quad \vec{v} \cdot d\vec{a} = -2yz dx dy = 0 \Rightarrow \int = 0,
\end{aligned}$$

and the sum over the 6 faces is  $\frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 = 2$ , as for the left hand side!

#### 6.1.4 Fundamental theorem for curls: Stokes theorem

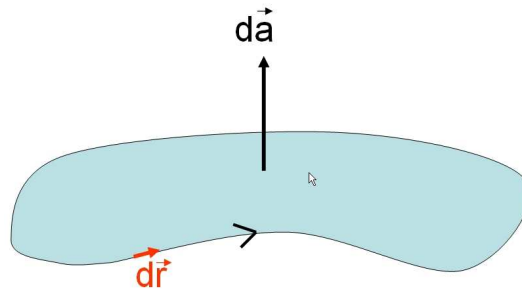


Figure 6: Directed area measure is perpendicular to loop according to right hand rule.

The proof here is very similar to the divergence theorem, so I won't belabor it here (I recommend you read Boaz or Shankar), but simply state it. In this case we consider a directed closed path in space, the boundary of a 2D area  $A$ .

$$\int_A (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_{\partial A} \vec{v} \cdot d\vec{r}. \quad (11)$$

Remarks (Stokes and divergence thm):

1. Define the normal to the area according to the right hand rule using the sense of the loop, as in Fig. 6.

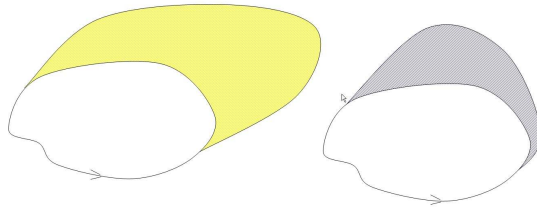


Figure 7: The same loop can be the boundary for different open surfaces.

2. Many surfaces can have the same boundary, see Fig. 10
3. If  $\vec{v} = \vec{\nabla} \times \vec{B}$ , then

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \Rightarrow \oint (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = 0 \quad (12)$$

since

$$\int_V \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) d\tau = \int_{\partial V} (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} \quad (13)$$

4. If  $\vec{v} = \vec{\nabla} \phi$ , then  $\vec{\nabla} \times \vec{v} = 0$ , so  $\oint \vec{v} \cdot d\vec{r} = 0$ .  $\vec{v}$  is conservative.

Some tricks:

1. If you need  $\int_A^B \vec{F} \cdot d\vec{r}$  along a path between A and B, check to see if  $\vec{F}$  is conservative (Do you know or can you check that  $\vec{F} = \vec{\nabla} \phi$  or  $\vec{\nabla} \times \vec{F} = 0$ ?) If yes, choose the simplest path!

2.

$$\int_{A_1} (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{r} = \int_{A_2} (\vec{\nabla} \times \vec{v}) \cdot d\vec{a}, \quad (14)$$

if  $A_1$  and  $A_2$  are bounded by same loop—choose simplest  $A$ .

3. If  $\vec{\nabla} \cdot \vec{v} = 0$ , then  $\int_A \vec{v} \cdot d\vec{a} = \int_{A_1} \vec{v} \cdot d\vec{a} + \int_{A_2} \vec{v} \cdot d\vec{a} = 0$ , so if you can do  $\int_{A_1} \vec{v} \cdot d\vec{a}$  you can find  $\int_{A_2} \vec{v} \cdot d\vec{a}$ .

### 6.1.5 Intuition for vector fields

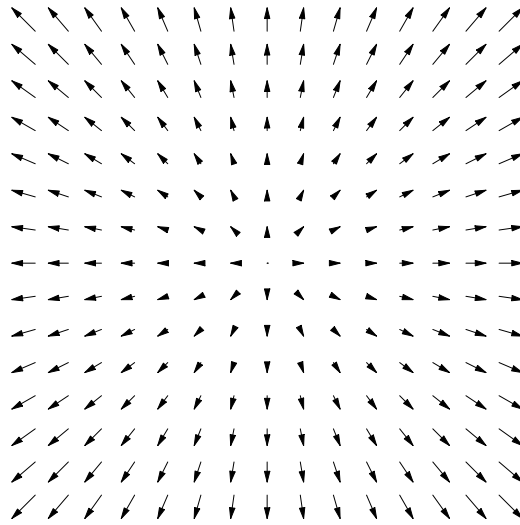


Figure 8: Example 1. “Diverging” radial field  $\vec{v} = \vec{r} = (x, y)$ .  $\vec{\nabla} \cdot \vec{v} = 3 > 0$ , but  $\vec{\nabla} \times \vec{v} = 0$ . Vector field “spreads out” or “diverges”. Flux  $\int \vec{v} \cdot d\vec{a}$  through closed surface surrounding  $\vec{r} = 0$  obviously nonzero.

*Remarks on divergenceless fields.* If  $\vec{\nabla} \cdot \vec{v} = 0$  everywhere, the following conditions are equivalent:

1.  $\int_{\text{closed surface}} \vec{v} \cdot d\vec{a} = 0$
2.  $\vec{v}$  is the curl of some vector,  $\vec{v} = \vec{\nabla} \times \vec{w}$ . (Why?) Note it is obviously not unique, we can  $\vec{\nabla} \phi$  to  $\vec{w}$  for any scalar field  $\phi$ , and still maintain  $\vec{v} = \vec{\nabla} \times \vec{w}$ . This might remind you of “gauge symmetry” in electromagnetism.
3.  $\int_{\text{open surface}} \vec{v} \cdot d\vec{a}$  is independent of the surface for a given boundary line.

*Remarks on curlless (conservative) fields.* If  $\vec{\nabla} \times \vec{v} = 0$  everywhere,

1.  $\oint \vec{v} \cdot d\vec{r} = 0$  by Stokes
2.  $\vec{v}$  is the gradient of some scalar  $\vec{v} = -\vec{\nabla} \phi$
3.  $\int_a^b \vec{v} \cdot d\vec{r}$  independent of path between  $a$  and  $b$ .



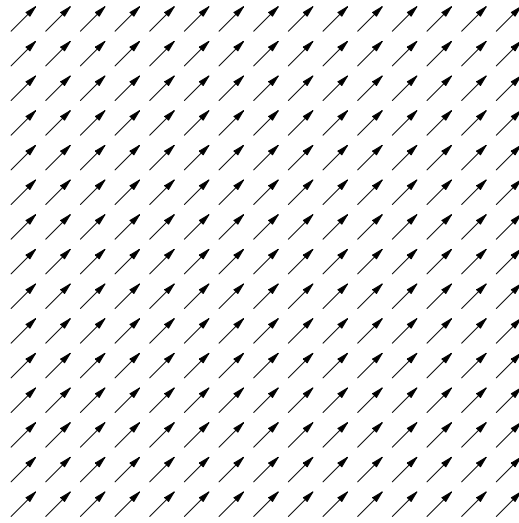


Figure 9: Example 2. Constant vector field  $\vec{v} = (1, 1)$ .  $\vec{\nabla} \cdot \vec{v} = 0$ . No spreading. Net flux through any closed surface is zero.

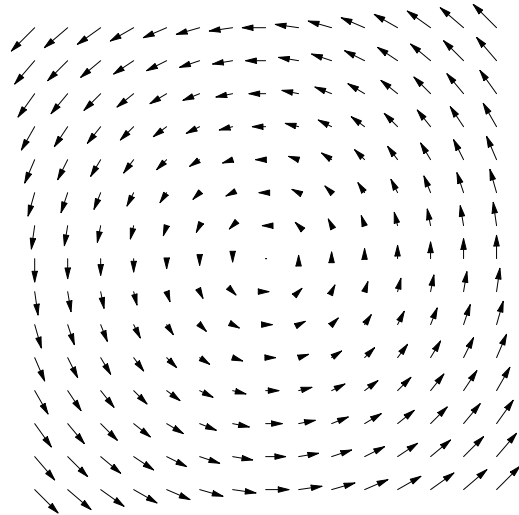


Figure 10: Ex. 3.  $\vec{v} = (-y, x, 0)$   $\vec{\nabla} \cdot \vec{v} = 0$ , but  $\vec{\nabla} \times \vec{v} = (0, 0, 2) \neq 0$ . Line integral  $\oint \vec{v} \cdot d\vec{r} = \oint r\hat{\theta} \cdot r\hat{\theta}d\theta = 2\pi r^2$ . Compare with integral of curl dotted into surface, take cylinder for simplicity:  $\vec{\nabla} \times \vec{v} \cdot d\vec{a} = 2 \cdot \pi r^2$ . Stokes works!

### 6.1.6 Divergence of Coulomb or gravitational force of point charge/mass

Consider the vector field  $\vec{v}$  with the form

$$\vec{v} = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3}, \quad (15)$$

i.e. it points radially outward and falls off as  $1/r^2$ . This should remind you of the electric field of a point charge, or the gravitational field of a point mass. Here's a little paradox:

claim:  $\vec{\nabla} \cdot \vec{v} = 0$ :

$$\vec{\nabla} \cdot \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} = \left( \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} \right) \quad (16)$$

$$+ (x \leftrightarrow y) + (x \leftrightarrow z) \quad (17)$$

$$= \frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} + (x \leftrightarrow y) + (x \leftrightarrow z) \quad (18)$$

$$= 0. \quad (19)$$

OK, so this field is divergenceless. It looks like something is flowing out, but on the other hand the magnitude of  $\vec{v}$  is getting smaller as you go out, so maybe it's ok. BUT what about the divergence theorem? If I calculate

$$\int_{\text{sphere}} \vec{v} \cdot d\vec{a} = \int \frac{\hat{r}}{r^2} \cdot \hat{r} r^2 \sin \theta d\theta d\phi = 4\pi. \quad (20)$$

So, is the divergence theorem wrong, or did we miss something? The answer is that we were a little careless in calculating the divergence and declaring it to be zero. It is zero, almost everywhere. However, if you look at our equations exactly at the point  $x = y = z = 0$ , you will see that the derivatives diverge, i.e. are infinite (you may need to do L'Hospital's rule to convince yourself, but it's right). So we have encountered a very peculiar situation, a function of space which is infinite at one point, but zero everywhere else. This type of function<sup>2</sup> was explored in the physics context by Paul Dirac, so we refer to it as the Dirac delta-function, and give it the symbol  $\delta$ , *defined* as

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}, \quad \text{such that } \int_{-\infty}^{\infty} \delta(x) dx = 1, \quad (21)$$

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<sup>2</sup>mathematically, it is not a function; it's called a distribution.

and

$$\delta^{(3)}(\vec{r}) = \delta(x)\delta(y)\delta(z). \quad (22)$$

So since

$$\int_{\tau} \vec{\nabla} \cdot \frac{\hat{r}}{r^2} = \int_{\partial\tau} \frac{\hat{r}}{r^2} \cdot d\vec{a} = 4\pi, \quad (23)$$

we have determined that

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r}). \quad (24)$$

Alternate definitions of  $\delta(x)$  you will see:

define a sequence of “normalized” functions, e.g.

$$\delta_n(x) = \begin{cases} 0 & |x| > \frac{1}{2n} \\ n & |x| < \frac{1}{2n} \end{cases} \quad \text{or} \quad \delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}, \quad (25)$$

for which one can show that

$$\int_{-\infty}^{\infty} \delta_n(x) dx = 1, \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0), \quad (26)$$

for *any* function  $f(x)$ . Then define  $\delta(x)$  as

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx \equiv \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0). \quad (27)$$

Note  $\lim_{n \rightarrow \infty} \delta_n(x)$  does not exist, but integral does. That’s what makes  $\delta(x)$  a distribution, not a function, according to mathematicians.

### 6.1.7 Properties of $\delta$ -function

- $\delta(x) = 0$  if  $x \neq 0$ .
- $\int_{-\infty}^{\infty} \delta(x - a) f(x) dx = f(a)$
- $\delta(ax) = \frac{1}{|a|} \delta(x)$

$$\delta(g(x)) = \sum_{a_n} \frac{\delta(x - a_n)}{|g'(a_n)|}, \quad (28)$$

where  $a_n$  are the zeros of  $g(x) = 0$ .

Ex. 1:

$$\int_0^5 x^3 \delta(x - 3) dx = 3^3 \quad \text{but} \quad \int_0^2 x^3 \delta(x - 3) dx = 0. \quad (29)$$

Ex. 2:

$$\int_{-\infty}^{\infty} (x + 1)^3 \delta(5x) dx = \frac{1}{5} \int_{-\infty}^{\infty} (x + 1)^3 \delta(x) dx = \frac{1}{5} \quad (30)$$

Ex. 3:

$$\int_0^{\infty} x^3 \delta(x^2 - 1) dx = \int_0^{\infty} x^3 \left[ \frac{\delta(x - 1)}{|g'(1)|} + \frac{\delta(x + 1)}{|g'(-1)|} \right] dx \quad (31)$$

$$= \frac{1}{2} \left[ \int_0^{\infty} x^3 \delta(x - 1) dx + \int_0^{\infty} x^3 \delta(x + 1) dx \right] \quad (32)$$

$$= \frac{1}{2}(1 + 0) = \frac{1}{2} \quad (33)$$