9 Linear algebra

Read: Boas Ch. 3.

9.1 Properties of and operations with matrices

$M \times N$ matrix with elements A_{ij}

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1j} & \ldots & A_{1N} \\ A_{21} & A_{22} & \ldots & A_{2j} & \ldots & A_{2N} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{i1} & A_{i2} & \ldots & A_{ij} & \ldots & A_{iN} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{M1} & A_{M2} & \ldots & A_{Mj} & \ldots & A_{MN} \end{bmatrix} \tag{1}$$

Definitions:

- Matrices are equal if their elements are equal, $A = B \Leftrightarrow A_{ij} = B_{ij}$.
- $(A + B)_{ij} = A_{ij} + B_{ij}$
- $(kA)_{ij} = kA_{ij}$ for k const.
- $(AB)_{ij} = \sum_{\ell=1}^{N} A_{i\ell} B_{\ell j}$. Note for multiplication of rectangular matrices, need $(M \times N) \cdot (N \times P)$.
- Matrices need not "commute". AB not nec. equal to BA. $[A, B] \equiv AB - BA$ is called "commutator of A and B. If $[A, B] = 0$, A and B commute.
- For square mats. $N \times N$, $\det A = |A| = \sum_{\pi} \text{sgn}\pi \, A_{1\pi(1)} A_{2\pi(2)} \ldots A_{N\pi(N)}$, where sum is taken over all permutiations π of the elements $\{1, \ldots N\}$. Each term in the sum is a product of N elements, each taken from a different row of A and from a different column of A, and sgn π. Examples:

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}, \tag{2}$$

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} - A_{11}A_{22}A_{32} \tag{3}$$

- $\det AB = \det A \cdot \det B$ but $\det(A + B) \neq \det A + \det B$. For practice with determinants, see Boas.

- Identity matrix I: $IA = A \ \forall A$. $I_{ij} = \delta_{ij}$.
- Inverse of a matrix. $A \cdot A^{-1} = A^{-1}A = I$.
- Transpose of a matrix $(A^T)_{ij} = A_{ji}$.
- Formula for finding inverse:

$$A^{-1} = \frac{1}{\det A} C^T, \tag{4}$$

  where C is "cofactor matrix". An element C_{ij} is the determinant of the $N-1 \times N-1$ matrix you get when you cross out the row and column (i,j), and multiply by $(-1)^i(-1)^j$. See Boas.
- Adjoint of a matrix. $A^\dagger$ is adjoint of A, has elements $A^\dagger_{ij} = A^*_{ji}$, i.e. it's conjugate transpose. Don't worry if you don't know or have forgotten what conjugate means.
- $(AB)^T = B^T A^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- "row vector" is $1 \times N$ matrix: $[a \ b \ c \ldots \ n]$
- "column vector" is $M \times 1$ matrix:

$$\begin{bmatrix} e \\ f \\ g \\ \vdots \\ m \end{bmatrix} \tag{5}$$

- Matrix is "diagonal" if $A_{ij} = A_{ii}\delta_{ij}$.
- "Trace" of matrix is sum of diagonal elements: Tr $A = \sum_i A_{ii}$. Trace of produce is invariant under cyclic permutations (check!):

$$\text{Tr } ABC = \text{Tr } BCA = \text{Tr } CAB. \tag{6}$$

9.2 Solving linear equations

Ex.

$$\begin{aligned} x - y + z &= 4 \\ 2x + y - z &= -1 \\ 3x + 2y + 2z &= 5 \end{aligned}$$

may be written

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}. \tag{7}$$

Symbolically, $A \cdot \vec{r} = \vec{k}$. What we want is $\vec{r} = A^{-1}\vec{k}$. So we find the determinant $\det A = 12$, and the cofactor matrix

$$C = \begin{bmatrix} 4 & -7 & 1 \\ 4 & -1 & -5 \\ 0 & 3 & 5 \end{bmatrix}, \tag{8}$$

then take the transpose and construct $A^{-1} = \frac{1}{\det A} C^T$:

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 4 & 4 & 0 \\ -7 & -1 & 3 \\ 1 & -5 & 5 \end{bmatrix}, \tag{9}$$

so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 4 & 4 & 0 \\ -7 & -1 & 3 \\ 1 & -5 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}. \tag{10}$$

So $x = 1$, $y = -1$, $z = 2$. Alternate method is to use Cramer's rule (see Boas p. 93):

$$x = \frac{1}{12} \begin{vmatrix} 4 & -1 & 1 \\ -1 & 1 & -1 \\ 5 & 2 & 2 \end{vmatrix} = 1, \tag{11}$$

and "similarly" for y and z. Here, the 12 is $\det A$, and the determinant shown is that of the matrix of coefficients A with the x coefficients (in this case) replaced by $\vec{k}$.

Q: what happens if $\vec{k} = 0$? Eqs. are *homogeneous* $\Rightarrow \det A = 0$.

9.3 Rotation matrices

Here's our old example of rotating coordinate axes. $\vec{r} = (x, y)$ is a vector. Let's call $\vec{r}' = (x', y')$ the vector in the new coordinate system. The two are related by

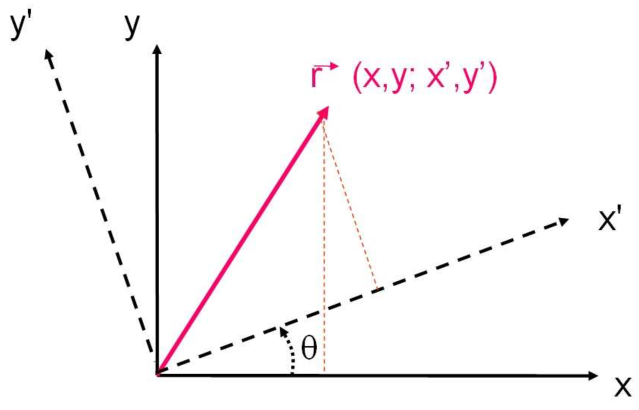

Figure 1: Rotation of coordinate axes by θ.

the equations of coordinate transformation we discussed in week 4 of the course. These may be written in matrix form in a very convenient way(check):

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\vec{r}\,'} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{R_\theta} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{r},} \tag{12}$$

where R_θ is a *rotation matrix* by θ. Note the transformation preserves lengths of vectors $|\vec{r}\,'| = |\vec{r}|$ as we mentioned before. This means the rotation matrix is *orthogonal*:

$$R_\theta^T R_\theta = I. \tag{13}$$

These matrices have a special property ("group property"), which we can show by doing a second rotation by θ':

$$R_\theta R_{\theta'} = \begin{bmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta'\cos\theta - \sin\theta'\sin\theta & \cos\theta'\sin\theta + \sin\theta'\cos\theta \\ -\sin\theta'\cos\theta - \cos\theta'\sin\theta & \cos\theta'\cos\theta - \sin\theta'\sin\theta \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} \cos(\theta+\theta') & \sin(\theta+\theta') \\ -\sin(\theta+\theta') & \cos(\theta+\theta') \end{bmatrix} = R_{\theta+\theta'}. \tag{15}$$

Thus the transformation is *linear*. More general def. $A(c_1B+c_2D) = c_1AB+c_2AD$.

9.4 Matrices acting on vectors

$$\begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ \\ \vdots \\ v_N' \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1j} & ... & A_{1N} \\ A_{21} & A_{22} & ... & A_{2j} & ... & A_{2N} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{i1} & A_{i2} & ... & A_{ij} & ... & A_{iN} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{N1} & A_{N2} & ... & A_{Nj} & ... & A_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ \\ \vdots \\ v_N \end{bmatrix}, \tag{16}$$

or more compactly

$$v_i' = \sum_{j=1}^{N} A_{ij} v_j. \tag{17}$$

Another notation you will often see comes from the early days of quantum mechanics. Write the same equation

$$|v'\rangle = A|v\rangle. \tag{18}$$

So this is a "ket", or column vector. A "bra", or row vector is written as the *adjoint* of the column vector[1]:

$$\langle v| = (|v\rangle)^\dagger \equiv (v_1^* \; v_2^* \; \ldots \; v_N^*) \tag{19}$$

$$= (v_1 \; v_2 \; \ldots \; v_N) \quad \text{if } v_i \text{ are real.} \tag{20}$$

N.B. In this notation the scalar product of $|v\rangle$ and $|w\rangle$ is $\langle v|w\rangle$, and the length of a vector is given by $|v|^2 = \langle v|v\rangle$.

9.5 Similarity transformations

Suppose a matrix B rotates $|r\rangle$ to $|r_1\rangle$, $|r_1\rangle = B|r\rangle$. Now we rotate the coordinate system by some angle as well, so that the vectors in the new system are $|r'\rangle$ and $|r_1'\rangle$, e.g. $|r_1'\rangle = R|r_1\rangle$ What is the matrix which relates $|r'\rangle$ to $|r_1'\rangle$, i.e. the transformed matrix B in the new coordinate system?

$$|r_1'\rangle = R|r_1\rangle = RB|r\rangle = RBR^{-1}R|r\rangle = (RBR^{-1})(R|r\rangle) = (RBR^{-1})|r'\rangle, \tag{21}$$

so the matrix B in the new basis is

$$B' = RBR^{-1}. \tag{22}$$

[1] The name comes from putting bra and ket together to make the word bracket; no one said physicists were any good at language

This is called a similarity transformation of the matrix B.

To retain:

- similarity transformations preserve traces and determinants: Tr $M =$ Tr M', $\det M = \det M'$.
- matrices R which preserve lengths of real vectors are called *orthogonal*, $RR^T = 1$ as we saw explicitly.
- matrices which preserve lengths of *complex* vectors are called *unitary*. Suppose $|v'\rangle = U|v\rangle$, require $UU^\dagger = 1$, then

$$\langle v'|v'\rangle = (U|v\rangle)^\dagger U|v\rangle = \langle v|U^\dagger U|v\rangle = \langle v|v\rangle. \tag{23}$$

  A similarity transformation with unitary matrices is called a unitary transformation.

- If a matrix is equal to its adjoint, it's called self-adjoint or *Hermitian*.

Examples

1.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \tag{24}$$

is *symmetric*, i.e. $A = A^T$, also Hermitian because it is real.

2.

$$A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix} \tag{25}$$

is *antisymmetric*, and anti-self-adjoint, since $A = -A^T = -A^\dagger$.

3.

$$A = \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix} \tag{26}$$

is *Hermitian*, $A = A^\dagger$.

4.

$$A = \begin{bmatrix} i & 1 \\ -1 & 2i \end{bmatrix} \tag{27}$$

is *antiHermitian*, $A = -A^\dagger$. Check!

5.

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \tag{28}$$

is *unitary*, $A^{-1} = A^\dagger$.

6.

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{29}$$

is orthogonal, $A^{-1} = A^T$. Check!

9.5.1 Functions of matrices

Just as you can define a function of a vector (like $\vec{r}!$), you can define a function of a matrix M, e.g. $F(M) = aM^2 + bM^5$ where a and b are constants. The interpretation here is easy, since powers of matrices can be understood as repeated matrix multiplication. On the other hand, what is $\exp(M)$? It must be understood in terms of its Taylor expansion, $\exp(M) = \sum_n M^n/n!$. Note that this makes no sense unless *every* matrix element sum converges.

Remark: note $e^A e^B \neq e^{A+B}$ unless $[A, B] = 0$! Why? (Hint: expand both sides.)