PHY3113–Introduction to Theoretical Physics

Fall 2008

Test 1 SOLUTIONS

Oct. 1, 2008

1. Calculate the Joule-Thompson coefficient $\left(\frac{\partial u}{\partial v}\right)_T$, where u is the internal energy and v is the volume, for a gas with equation of state $p = RT/(v-b) - a/v^2$. [Hint: use $du = Tds - pdv$ and Maxwell relation $\left(\frac{\partial s}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$.]

Start with $du = Tds - pdv$ and $p = RT/(v-b) - a/v^2$. Consider $s = s(v,T)$. Then

$$du = T\left[\left(\frac{\partial s}{\partial V}\right)_T dv + \left(\frac{\partial s}{\partial T}\right)_v dT\right] - pdv. \tag{1}$$

Now we see that the derivative requested is

$$\left(\frac{\partial u}{\partial v}\right)_T = T\left(\frac{\partial s}{\partial v}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_v - p = T\frac{R}{v-b} - p = \frac{a}{v^2} \tag{2}$$

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2. Consider the triangle in the (x,y) plane with vertices at (-1,0), (1,0), and (0,1). Evaluate the closed line integral

$$I = \oint (-y\hat{x} + x\hat{y}) \cdot d\vec{r} \tag{3}$$

around the boundary of the triangle in the anticlockwise direction.

$d\vec{r} = \hat{x}dx + \hat{y}dy$, so $(-y\hat{x} + x\hat{y}) \cdot d\vec{r} = -ydx + xdy$.

On leg (-1,0) $\rightarrow$ (1,0) we have $y = 0$, so integral is $\int_{-1}^{1}(-y)dx = 0$. On the leg (1,0) $\rightarrow$ (0,1) we have $y = -x+1$, so integral is -$\int_1^0(-x+1)dx + \int_0^1(1-y)dy = \frac{1}{2} + \frac{1}{2} = 1$. On the path (0,1) $\rightarrow$ (-1,0) we have $y = x+1$, so integral is $-\int_0^{-1}(x+1)dx + \int_1^0(y-1)dy = \frac{1}{2} + \frac{1}{2} = 1$. So total line integral is 2.

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3. Consider the parabola $y = 4 + 5x^2$. Find the closest point to the origin on this curve by the method of Lagrange multipliers.

I actually did it 3 ways to illustrate the possibilities:

(a) substituting explicitly into the distance formula for $y(x)$ and solve the conventional 1D minimization problem.

(b) substituting explicitly into the distance formula for $x(y)$ and solve the conventional 1D minimization problem.

(c) using the method of Lagrange multipliers.

We'll minimize $x^2 + y^2$ rather than $\sqrt{x^2 + y^2}$ as usual:

(a) $y = 4 + 5x^2$ so $x^2 + y^2 = x^2 + (4 + 5x^2)^2$. Minimize

$$\frac{d}{dx}\left(25x^4 + 41x^2 + 16\right) = 100x^3 + 82x = 0 \quad \Rightarrow \quad x = 0,\ y = 4 \quad \checkmark \tag{4}$$

(b) $x^2 = (y - 4)/5$, so minimize $(y - 4)/5 + y^2$:

$$\frac{d}{dy}(y - 4)/5 + y^2 = \frac{1}{5} + 2y = 0 \quad \Rightarrow \quad y = -\frac{1}{10}. \tag{5}$$

But this value cannot lie on the parabola, so it must be spurious somehow. Going back, we see that at this value of y the quantity x^2 on the parabola becomes negative, so this is not a valid solution for a point x, y on the parablola. The minimum must take place on the boundary of the set of x, y lying on the parabola, i.e. $y = 4$, implying $x = 0$.

(c) Take $f = x^2 + y^2$, function to be minimized in unconstrained space according to M. Lagrange is

$$F = f + \lambda(4 + 5x^2 - y) \tag{6}$$

So 3 equations for a minimum are

$$\frac{\partial F}{\partial x} = 0 = 2x + 10x\lambda \ ; \quad \frac{\partial F}{\partial y} = 0 = 2y - \lambda \ ; \quad \frac{\partial F}{\partial \lambda} = 0 = 4 + 5x^2 - y. \tag{7}$$

1st equation admits a solution $x = 0$ or $\lambda = -1/5$. The first one is correct, yields $y = 4$ from constraint (3rd) equation. Second one gives $y = -1/10$ again, this is the spurious solution discussed above.

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4. Calculate the total derivative dy/dx for $x = \frac{y-2}{y+4}$ in two ways:

(a) (4 pts.) explicitly solve for $y(x)$

(b) (4 pts.) use implicit differentiation.

(c) (2pts.) Verifiy that your answer is the same in both cases.

(a) Solve by finding $y(x)$, $y = (4x + 2)/(1 - x)$, so $dy/dx = 6/(1 - x)^2$.

(b) Implicitly:

$$\begin{aligned} x(y + 4) &= y - 2 \quad \Rightarrow \quad dx(y + 4) + xdy = dy \\ &\Rightarrow dy = \frac{dx(y + 4)}{1 - x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y + 4}{1 - x} \end{aligned} \tag{8}$$

(c)

$$\frac{4 + y}{1 - x} = \frac{4 + \frac{4x+2}{1-x}}{1 - x} = \frac{6}{(1 - x)^2} \tag{9}$$

5. Using the properties of the Levi-Civita symbol, verify the vector identity

$$\vec{A} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{2}\vec{\nabla}(A^2) - (\vec{A} \cdot \vec{\nabla})\vec{A} \tag{10}$$

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repeated summation index convention:

$$\begin{aligned}\left(\vec{A} \times (\vec{\nabla} \times \vec{A})\right)_i &= \epsilon_{ijk} A_j (\vec{\nabla} \times \vec{A})_k = \epsilon_{ijk} A_j \epsilon_{k\ell m} \nabla_\ell A_m = (\epsilon_{ijk}\epsilon_{\ell mk}) A_j \nabla_\ell A_m \\ &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) A_j \nabla_\ell A_m = \frac{1}{2}\nabla_i A^2 - (\vec{A} \cdot \vec{\nabla}) A_i \end{aligned} \tag{11}$$

Note that I used $A_j \nabla_i A_j = \frac{1}{2}\nabla_i A^2$, just the product rule.

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6. (Extra credit, 5 pts.) In the integral

$$I = \int_{x=0}^{1/2} \int_{y=x}^{1-x} \left(\frac{x-y}{x+y}\right)^2 dydx, \tag{12}$$

make the transformation

$$x = \frac{1}{2}(r - s) \quad ; \quad y = \frac{1}{2}(r + s), \tag{13}$$

and evaluate I. [Hint: sketch the area of integration in $x - y$ plane, then draw the r and s axes. Determine the area of $r - s$ integration.]

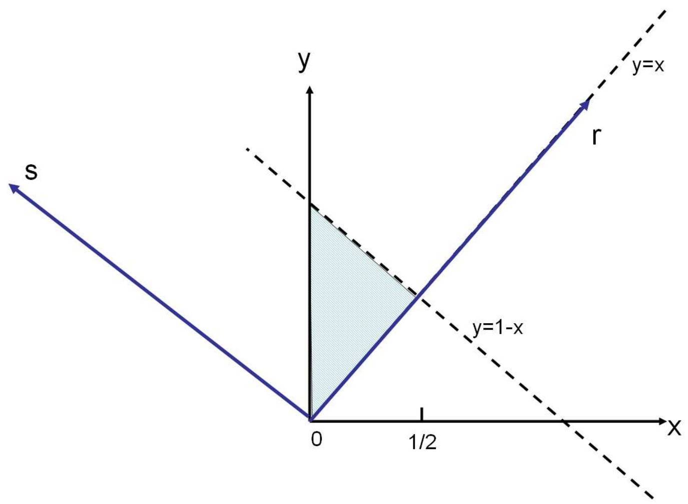

FIG. 1: Variables x, y transformation to r, s.

Jacobian of inverse transformation $r = x + y$, $s = y - x$ is

$$J\left(\frac{x,y}{r,s}\right) = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}, \tag{14}$$

so

$$I = \int_0^1 dr \int_0^r ds \left(\frac{s}{r}\right)^2 \cdot \frac{1}{2} = \int_0^1 \frac{1}{r^2} dr \left(\frac{1}{2}\frac{s^3}{3}\Big|_0^r\right) = \int_0^1 \frac{r}{6} dr = \frac{1}{12}. \tag{15}$$

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7. (Extra credit, 5 pts.) Planck's theory of quantized oscillators led to an average energy

$$\langle\epsilon\rangle = \frac{\sum_{n=1}^{\infty} n\epsilon_0 \exp(-n\epsilon_0/kT)}{\sum_{n=0}^{\infty} \exp(-n\epsilon_0/kT)}, \tag{16}$$

where ϵ_0 was a constant energy. Find $d\langle\epsilon\rangle/dT$ in closed form (evaluate all sums).

First call $\alpha = \epsilon_0/kT$. Then note that $\sum_n \exp(-\alpha n) = (1 - \exp(-\alpha))^{-1}$ is the sum of the geometric series, and that

$$\frac{d}{d\alpha}\sum_n \exp(-\alpha n) = -\sum_n n \exp(-\alpha n)$$

and

$$\frac{d}{d\alpha}\sum_n \exp(-\alpha n) = \frac{d}{d\alpha}\left(\frac{1}{1-e^{-\alpha}}\right) = -\frac{e^{-\alpha}}{(1-e^{-\alpha})^2} \tag{17}$$

So

$$\langle\epsilon\rangle = -\epsilon_0 \frac{e^{-\alpha}}{1-e^{-\alpha}} = \frac{\epsilon_0}{e^{\alpha}-1}. \tag{18}$$

So

$$\frac{d\langle\epsilon\rangle}{dT} = \frac{d\langle\epsilon\rangle}{d\alpha}\frac{d\alpha}{dT} = \epsilon_0 \cdot \frac{\epsilon_0}{k} \cdot \frac{-1}{T^2}\frac{e^{\alpha}}{(e^{\alpha}-1)^2} = k\alpha^2 \frac{e^{\alpha}}{(e^{\alpha}-1)^2}. \tag{19}$$
