PHY3113–Introduction to Theoretical Physics

Fall 2007

Test 2 – 55 minutes

Oct. 26, 2007

No other materials allowed. If you can't do one part of a problem, solve subsequent parts in terms of unknown answer–define clearly. **Do 4 of 6 problems, clearly indicating which you want graded!**. *You may attempt the remaining two as extra credit as well. All regular parts are worth 10 pts., each extra credit 5 each, for maximum of 50 points.*

Useful formulae:

$$\vec{\nabla}\psi = \frac{\partial \psi}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\,\hat{\phi}.$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\begin{aligned} x &= r\sin\theta\cos\phi = \rho\cos\vartheta \\ y &= r\sin\theta\sin\phi = \rho\sin\vartheta \\ z &= r\cos\theta = z \end{aligned}$$

$$\begin{aligned} \nabla\cdot\vec{E} &= \rho/\epsilon_0 \\ \nabla\cdot\vec{B} &= 0 \\ \nabla\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla\times\vec{B} &= \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \end{aligned}$$

$$\begin{aligned} \vec{E} &= -\nabla\Phi \\ \vec{B} &= \nabla\times\vec{A} \end{aligned}$$

$$\vec{\nabla}\psi = \sum_i \hat{q}_i\,\frac{1}{h_i}\frac{\partial \psi}{\partial q_i}.$$

1. (a-c) For each matrix A, calculate $A^\dagger$, $\det A$ and A^{-1}. On this basis, state whether each is Hermitian, unitary, and/or orthogonal, ...:

(a)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

(b)

$$\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix} \tag{2}$$

(c)

$$\begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{3}$$

(d) Solve the system of equations

$$\begin{bmatrix} 8 & -2/3 \\ -4 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4}$$

explicitly by Cramer's rule.

2. Evaluate

(a)

$$\int_{-\infty}^{\infty} dx\, \delta(2x - \pi) \sin x \tag{5}$$

(b)

$$\int_{\pi/4}^{3\pi/4} d\theta\, \delta(\cos\theta)(\sin^2\theta + 1) \tag{6}$$

(c)

$$\int d\tau\ \delta(r - a) \tag{7}$$

(note $d\tau$ is 3D volume measure d^3r)

(d) What is the charge density $\rho(x, y, z)$ for an infinitely long but infinitesimally thick line of charge, with charge per unit length λ, oriented parallel to the x axis passing through the point (0,1,0)?

3. Given a vector field $\vec{F} = r\cos\theta\hat{r} + 2r\hat{\theta} + 2r\cos\theta\sin\phi\,\hat{\phi}$, calculate

(a) $\vec{\nabla} \cdot \vec{F}$.

(b) $\int_\tau \vec{\nabla} \cdot \vec{F}\, d\tau$, where τ is the sphere of radius 1.

(c) $\int_A \vec{F} \cdot d\vec{a}$, where A is the surface of the sphere in (b).

(d) Verify Gauss's theorem comparing b) and c).

4. A magnetic vector potential is given by $\vec{A} = B_0 x\hat{y}$. Consider the cube of side a bounded by the origin and (a, a, a). Verify Stokes' theorem

$$\int B \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{r} \tag{8}$$

around the loop formed by the square in the xy plane which is one of the faces of this cube, by evaluating

(a) by first performing $\int \vec{B} \cdot d\vec{a}$,

(b) then performing $\oint \vec{A} \cdot d\vec{r}$ and checking that Eq. (8) holds.

Now suppose the magnetic field is increasing with time, $B_0(t) = \alpha t$, and that the cube is an open framework of wires on its edges. Each wire segment has resistance R.

(c) What is $\nabla \times \vec{E}$ generated in the space around the cube?

(d) The line integral of this electric field around the square in the xy plane, $\oint_\square \vec{E} \cdot d\vec{\ell}$, is the induced voltage, or EMF. What current flows in the wire square?

5. In the cylindrical coordinate system ρ, ϑ, z,

(a) Calculate the arc length ds^2 for the cylindrical coordinate system ρ, ϑ, z.

(b) Identify the scale factors h_i and write down the gradient of a scalar field ψ in this system.

(c) Write down an expression in terms of the unit vectors $\hat{\rho}$, $\hat{\vartheta}$, and $\hat{z}$, for the unit vector normal to the surface of a cone of height h and base radius r.

(d) Find expressions for $\hat{\rho}$, $\hat{\vartheta}$, and $\hat{z}$ in terms of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

6. A vector field has the form

$$\vec{A}(x,y,z) = (x + \alpha y, y + \beta x, z) \tag{9}$$

(a) Determine α and β such that $\nabla \times \vec{A} = 0$.

(b) Calculate the corresponding scalar potential Φ. Be sure to clearly specify all unknown constants.

(c) Sketch the vector field $\vec{A}$ in the plane $z = 0$ for $\alpha = 1$.

(d) Calculate the ratio of the line integrals

$$\frac{\int_{s_1} \vec{A} \cdot d\vec{r}}{\int_{s_2} \vec{A} \cdot d\vec{r}}, \tag{10}$$

where s_1 is the path along the diagonal between $a = (0,0,0)$ and $b = (1,1,1)$, and s_2 is the path $a \to (1,0,0) \to (1,1,0) \to b$.