PHY3113–Introduction to Theoretical Physics

Fall 2007

Test 2 SOLUTIONS

Oct. 26, 2007

Useful formulae:

$$\vec{\nabla}\psi = \frac{\partial \psi}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\,\hat{\phi}.$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\begin{aligned} x &= r\sin\theta\cos\phi = \rho\cos\vartheta \\ y &= r\sin\theta\sin\phi = \rho\sin\vartheta \\ z &= r\cos\theta = z \end{aligned}$$

$$\begin{aligned} \nabla\cdot\vec{E} &= \rho/\epsilon_0 \\ \nabla\cdot\vec{B} &= 0 \\ \nabla\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla\times\vec{B} &= \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \end{aligned}$$

$$\begin{aligned} \vec{E} &= -\nabla\Phi \\ \vec{B} &= \nabla\times\vec{A} \end{aligned}$$

$$\vec{\nabla}\psi = \sum_i \hat{q}_i\,\frac{1}{h_i}\frac{\partial \psi}{\partial q_i}.$$

1. For each matrix A, calculate $A^\dagger$, $\det A$ and A^{-1}. On this basis, state whether each is Hermitian, unitary, and/or orthogonal, ...:

   (a)

$$A^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \det A = -1,\ C^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (1)$$

$$A^{-1} = A^\dagger,\ \textit{unitary or orthogonal}, \quad (2)$$

$$A = A^\dagger,\ \textit{hermitian}. \quad (3)$$

(b)

$$
\begin{aligned}
A^\dagger &= \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}, \ \det A = 6-2 = 4, & (4) \\
C &= \begin{bmatrix} 3 & -(1-i) \\ -(1+i) & 2 \end{bmatrix}, \ C^T = \begin{bmatrix} 3 & -(1+i) \\ -(1-i) & 2 \end{bmatrix}, & (5) \\
A^{-1} &= \frac{1}{4}\begin{bmatrix} 3 & -(1+i) \\ -(1-i) & 2 \end{bmatrix} = \begin{bmatrix} 3/4 & (-1-i)/4 \\ -(1-i)/4 & 1/2 \end{bmatrix}. & (6)
\end{aligned}
$$

since $A = A^\dagger$, hermitian

(c)

$$
A^{-1} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}, \ A^{-1} = A^\dagger \tag{7}
$$

orthogonal or *unitary* (real). Note this is a rotation about the y axis by angle α, so inverse is just rotation by $-\alpha$.

(d) Solve the system of equations

$$
\begin{bmatrix} 8 & -2/3 \\ -4 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8}
$$

explicitly by Cramer's rule.

$$
\det \begin{bmatrix} 8 & -2/3 \\ -4 & 1/2 \end{bmatrix} = \frac{4}{3} \tag{9}
$$

so

$$
x = \frac{3}{4}\det \begin{bmatrix} 1 & -2/3 \\ 1 & 1/2 \end{bmatrix} = \frac{7}{8}
$$

$$
y = \frac{3}{4}\det \begin{bmatrix} 8 & 1 \\ -4 & 1 \end{bmatrix} = 9
$$

2. Evaluate

(a)

$$
\int_{-\infty}^{\infty} dx\, \delta(2x-\pi)\sin x = \frac{1}{2}\int_{-\infty}^{\infty} dx\, \delta(x-\frac{\pi}{2})\sin x = \frac{1}{2}. \tag{10}
$$

(b)

$$\int_{\pi/4}^{3\pi/4} d\theta\, \delta(\cos\theta)(\sin^2\theta + 1) = \int_{\pi/4}^{3\pi/4} dx\, \frac{\delta(\theta - \frac{\pi}{2})}{|\sin\theta|_{\theta=\pi/2}} (\sin^2\theta + 1) \;=\; 2. \tag{11}$$

(c)

$$\int d\tau\, \delta(r - a) \;=\; \int_0^\infty r^2 \delta(r - a)\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \;=\; 4\pi a^2. \tag{12}$$

(d) What is the charge density $\rho(x, y, z)$ for an infinitely long but infinitesimally thick line of charge, with charge per unit length λ, oriented parallel to the x axis passing through the point (0,1,0)?

We know it must be proportional to $\delta(y-1)\delta(z)$, and that when we integrate over y and z in a plane perpendicular to the wire, we should recover the full charge density per unit length along x. Thus A

$$\int dy \int dz \delta(y - 1)\delta(z) = \lambda, \tag{13}$$

so that

$$\rho = \lambda\delta(y - 1)\delta(z). \tag{14}$$

Note this has dimensions of charge/ 3D volume.

3. Given a vector field $\vec{F} = r\cos\theta\hat{r} + 2r\hat{\theta} + 2r\cos\theta\sin\phi\,\hat{\phi}$, calculate

(a)

$$\vec{\nabla} \cdot \vec{F} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(2r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial\,(2r\cos\theta\sin\phi)}{\partial\phi} \tag{15}$$

$$= 3\cos\theta \;+\; 2\cot\theta \;+\; 2\cot\theta\cos\phi. \tag{16}$$

(b) (volume is sphere radius 1)

$$\int_\tau \vec{\nabla} \cdot \vec{F}\, d\tau \;=\; \int_0^1 r^2\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, (3\cos\theta \;+\; 2\cot\theta \;+\; 2\cot\theta\cos\phi)$$

$$= \frac{1}{3} \times 0 = 0. \tag{17}$$

(c) (area is surface of sphere radius 1)

$$d\vec{a} \;=\; \sin\theta d\theta d\phi \hat{r}, \tag{18}$$

$$\vec{F} \cdot d\vec{a} \;=\; \sin\theta\cos\theta d\theta d\phi. \tag{19}$$

$$\int_A \vec{F} \cdot d\vec{a} \;=\; 2\pi\frac{1}{2}\sin^2\theta|_0^\pi \;=\; 0. \tag{20}$$

(d) So Gauss's theorem is satisfied by comparing b) and c).

4. (a)-(b) First notice that since $\vec{A}$ points along $\hat{y}$ and is $\propto x$, the only contribution to line integral is along segment parallel to y axis at $x = a$. $\vec{A}$ is const. along this segment, so integral is trivial. The curl of $\vec{A}$ which we need for Stokes' thm. is just the magnetic field $\vec{B}$, and taking the curl leads to a constant $\vec{B}$ field pointing along z, which is constant over the top face of the cube. Its dot product with the normals of all other faces is zero, so

$$\oint \vec{A}\cdot d\vec{r} = \int_0^a B_0 a\, dy = B_0 a^2. \tag{21}$$
$$\vec{B} = B_0\hat{k}, \tag{22}$$
$$\int B\cdot d\vec{a} = B_0 a^2 = \oint \vec{A}\cdot d\vec{r}. \tag{23}$$

Now suppose the magnetic field is increasing with time, $B_0(t) = \alpha t$, and that the cube is an open framework of wires on its edges. Each wire segment has resistance R.
(c) What is $\nabla\times\vec{E}$ generated in the space around the cube?

Here we need to use Faraday's eqn. $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$. The term on the right hand side of Faraday's law is $-\alpha$, so $\vec{\nabla}\times\vec{E} = -\alpha\hat{k}$.
(d) The line integral of this electric field around the square in the xy plane, $\oint_\Box \vec{E}\cdot d\vec{\ell}$, is the induced voltage, or EMF. What current flows in the wire square?

$$\oint \vec{E}\cdot d\vec{\ell} = \int \vec{\nabla}\times\vec{E}\cdot d\vec{a} = -\alpha a^2. \tag{24}$$

Ohm's law says $I = V/R = -\alpha a^2/(4R)$ (since segment resistances add in series), so $I = -\alpha a^2/(4R)$.

5. (a)-(b) Calculate the arc length ds^2 and scale factors h_i for the cylindrical coordinate system ρ, ϑ, z.

$$x = r\cos\vartheta\,,\ y = r\sin\vartheta\,,\ z = z\,, \tag{25}$$
$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial\vartheta}d\vartheta + \frac{\partial x}{\partial z}dz = \cos\vartheta dr - r\sin\vartheta d\vartheta, \tag{26}$$
$$dy = \sin\vartheta dr + r\cos\vartheta d\vartheta, \tag{27}$$
$$dz = dz, \tag{28}$$
$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\vartheta^2 + dz^2. \tag{29}$$
$$h_r = 1,\ h_\vartheta = r,\ h_z = 1, \tag{30}$$
$$\vec{\nabla}\psi = \sum \hat{q}_i \frac{1}{h_i}\frac{\partial\psi}{\partial q_i} = \hat{r}\frac{\partial\psi}{\partial r} + \frac{\hat{\vartheta}}{r}\frac{\partial\psi}{\partial\vartheta} + \hat{z}\frac{\partial\psi}{\partial z}. \tag{31}$$

(c)Write down an expression in terms of the unit vectors $\hat{\rho}$, $\hat{\vartheta}$, and $\hat{z}$, for the unit vector normal to the surface of a cone of height h and base radius r.

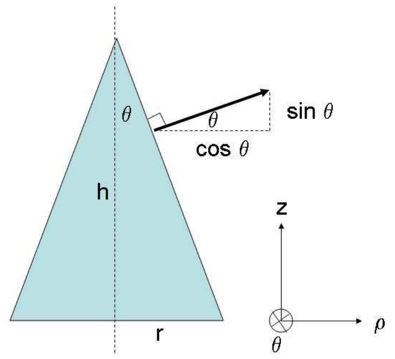

FIG. 1: cross section of cone.

Components of normal unit vector are shown, $\vartheta = \tan^{-1} r/h$, so $\hat{n} = \cos\vartheta\hat{\rho} + \sin\vartheta\hat{z}$.

(d) Find expressions for $\hat{\rho}$, $\hat{\vartheta}$, and $\hat{z}$ in terms of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

$$d\vec{s} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} \tag{32}$$

$$= d\rho\hat{\rho} + \rho d\vartheta\hat{\vartheta} + dz\hat{z} \tag{33}$$

Now we calculate $dx = \frac{\partial x}{\partial\rho}d\rho + \frac{\partial x}{\partial\vartheta}d\vartheta + \frac{\partial x}{\partial z}dz$, etc., and equate coefficients of $d\rho$, etc. to find

$$\begin{aligned}\hat{\rho} &= \hat{\imath}\cos\vartheta + \hat{\jmath}\sin\vartheta \\ \hat{\vartheta} &= -\hat{\imath}\sin\vartheta + \hat{\jmath}\cos\vartheta \\ \hat{z} &= \hat{k}\end{aligned} \tag{34}$$

6. A vector field has the form

$$\vec{A}(x, y, z) = (x + \alpha y, y + \beta x, z) \tag{35}$$

(a) Determine α and β such that $\nabla \times \vec{A} = 0$.

$$\begin{aligned}\vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x + \alpha y & y + \beta x & z \end{vmatrix} \\ &= \hat{k}(\beta - \alpha) = 0,\end{aligned}$$

so $\alpha = \beta$ works $\forall\alpha$.

(b) Calculate the corresponding scalar potential Φ. Be sure to clearly specify all unknown constants.

Using definition $\vec{A} = -\vec{\nabla}\Phi$, we have

$$\begin{aligned} \partial_x \Phi &= -x - \alpha y \\ \partial_y \Phi &= -y - \alpha x \\ \partial_z \Phi &= -z \end{aligned}$$

First condition $\Rightarrow \Phi = -x^2/2 - xy\alpha + c(y,z)$, second one $\Rightarrow \partial_y c(y,z) = -y \Rightarrow c(y,z) = -y^2 + c(z)$, third one $\Rightarrow \partial_z c(z) = -z \Rightarrow c(z) = -z^2/2 + c$. So

$$\Phi = -\frac{1}{2}(x^2 + y^2 + z^2) - \alpha xy + c \tag{36}$$

(c) Sketch the vector field $\vec{A}$ in the plane $z = 0$ for $\alpha = 1$.

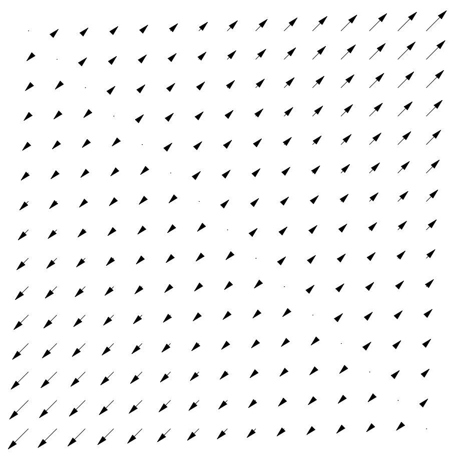

FIG. 2: Plot with $\alpha = 1$ with x, y running from -1 to 1.

(d) Calculate the ratio of the line integrals

$$\frac{\int_{s_1} \vec{A} \cdot d\vec{r}}{\int_{s_2} \vec{A} \cdot d\vec{r}}, \tag{37}$$

where s_1 is the path along the diagonal between $a = (0,0,0)$ and $b = (1,1,1)$, and s_2 is the path $a \to (1,0,0) \to (1,1,0) \to b$.

Without doing any calculations, we know the ratio is 1, since the field is conservative ($\nabla \times \vec{v} = 0$), so the line integral is independent of the path taken.