PHZ3113– to Theoretical Physics

Fall 2008

Test 2 – 55 minutes

Oct. 29, 2008

Solutions

Useful formulae:

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\,\hat{\phi}. \tag{1}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{2}$$

$$\begin{aligned}
\hat{i} &= \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \qquad & x = r\sin\theta\cos\phi \\
\hat{j} &= \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \qquad & y = r\sin\theta\sin\phi \\
\hat{k} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \qquad & z = r\cos\theta.
\end{aligned} \tag{3}$$

$$\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\frac{\widehat{\vec{r}-\vec{r}\,'}}{|\vec{r}-\vec{r}\,'|^2}, \tag{4}$$

where

$$\widehat{\vec{r}-\vec{r}\,'} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}. \tag{5}$$

$$\vec{\nabla}\psi = \sum_i \hat{q}_i\,\frac{1}{h_i}\frac{\partial\psi}{\partial q_i}. \tag{6}$$

1. Consider a vector field $\vec{v} = 2x\hat{x} - z\hat{y} + y\hat{z}$. Verify Stokes' theorem using the circle of radius a in the xy plane, bounding the surface A of the hemisphere above the (x, y) plane given by $x^2 + y^2 + z^2 = a^2$.

   (a) (5 pts) Calculate $\int_A \vec{\nabla}\times\vec{v}\cdot d\vec{a}$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2x & -z & y \end{vmatrix} = 2\hat{i} \tag{7}$$

$$= 2(\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}), \tag{8}$$

where in the last step I expressed the answer in spherical coordinates because we need the integral over a hemisphere, which has normal $\hat{r}$. The magnitude of the area element on the surface is $da = a^2 \sin\theta d\theta d\phi$. Therefore

$$\vec{\nabla} \times \vec{v} \cdot d\vec{a} = 2\sin\theta\cos\phi\, da = 2a^2 \sin^2\theta \cos\phi d\theta d\phi, \tag{9}$$

and the integral is

$$\int \vec{\nabla} \times \vec{v} \cdot d\vec{a} = \int_0^{\pi/2} a^2 \sin^2\theta d\theta \int_0^{2\pi} d\phi\, \cos\phi = 0 \tag{10}$$

because the $\cos\phi$ is integrated over a period.

(b) (5 pts) Calculate $\oint_{\partial A} \vec{v} \cdot d\vec{r}$.

Method 1): For the circular path in the xy plane, $d\vec{r} = ad\phi\hat{\phi}$, so we need to express $\vec{v}$ in terms of spherical coordinates:

$$\begin{aligned} \vec{v} &= (2a\sin\theta\cos\phi)(\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}) \\ &\quad -a\cos\theta(\sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}) \\ &\quad +a\sin\theta\sin\phi(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}). \end{aligned} \tag{11}$$

For the circle in the xy plane, $\theta = \pi/2$ is fixed. The $\hat{\phi}$ component of $\vec{v}$ is then $-2a\cos\phi\sin\phi - 0 + 0 = -2a\cos\phi\sin\phi$. So

$$\oint_{\partial A} \vec{v} \cdot d\vec{r} = \int_0^{2\pi} a\, d\phi\, (-2a\cos\phi\sin\phi) = 0 \tag{12}$$

Method 2): in Cartesian coordinates, $\vec{v} \cdot d\vec{r} = 2xdx - zdy + ydz$. Over the circle $dz = 0$, $z = 0$, and $dx = d(a\cos\phi) = -a\sin\phi d\phi$. So again

$$\oint_{\partial A} \vec{v} \cdot d\vec{r} = \int_0^{2\pi} a\, d\phi\, (-2a\cos\phi\sin\phi) = 0 \tag{13}$$

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2. Given the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & i \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \tag{14}$$

(a) (2 pts) calculate $[A, B]$

$$[A, B] \equiv AB - BA = \begin{bmatrix} 1 & 1-i \\ 1-i & -1 \end{bmatrix} \tag{15}$$

(b) (2 pts) calculate $A^\dagger BA$

$$\begin{pmatrix} 0 & i \\ i & -2i \end{pmatrix} \tag{16}$$

(c) (3 pts) solve the equation $C\vec{r} = \vec{k}$, where $\vec{r} = (x, y, z)$ and $\vec{k} = (-2, 1, 0)$, for x, y, z by matrix methods.

$$C^{-1} = \begin{bmatrix} \frac{2}{5} & 1 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 1 & -\frac{2}{5} \end{bmatrix}, \tag{17}$$

so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & 1 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 1 & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ 1 \\ \frac{3}{5} \end{bmatrix} \tag{18}$$

(d) (3 pts) which of the matrices is orthogonal? Why?

$B^T = B^{-1}$.

3. (a) (5 pts) Give an example of a family of functions δ_n which satisfy the conditions for their limit as $n \to \infty$ to be the Dirac δ-function, and state what these conditions are. If you can't define the functions mathematically, sketch them.

We want functions normalized to 1 whose width gets narrower as $n \to \infty$. We did a couple of examples in class, and there are many of them, but the simplest is the sequence of "step" or "window" functions:

$$\delta_n(x) = \begin{cases} 0 & |x| > \frac{1}{2n} \\ n & |x| < \frac{1}{2n} \end{cases} \tag{19}$$

(b) (5 pts) Evaluate $\int_0^\infty \delta(x^2 - 4)e^x \, dx$

$$\delta(x^2 - 4) = \frac{1}{|2x|}\bigg|_{x=2} \delta(x-2) + \frac{1}{|2x|}\bigg|_{x=-2} \delta(x+2), \tag{20}$$

but only the 1st root is contained in the integration range, so

$$\int_0^\infty \delta(x^2 - 4)e^x \, dx = \frac{1}{4} \int_0^\infty \delta(x-2)e^x \, dx = e^2/4 \tag{21}$$

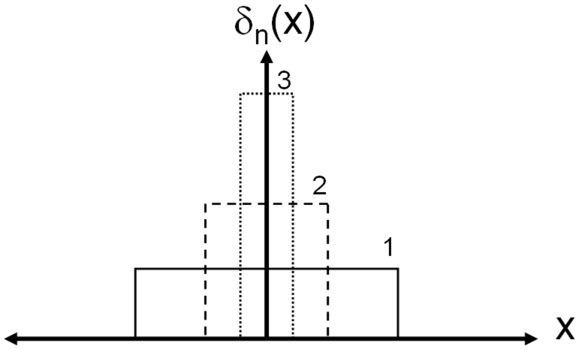

FIG. 1: Family $\delta_n(x)$.

4. Consider the transformations from coordinates (x, y) to (x', y') and from (x'', y'') to (x', y') given by

$$x' = (\sqrt{3}x + y)/2 \; ; \quad y' = (\sqrt{3}y - x)/2 \; ;$$
$$x' = (x'' + y'')/\sqrt{2} \; ; \quad y' = (y'' - x'')/\sqrt{2} \tag{22}$$

Find the transformation from (x, y) to (x'', y''). [Hint: write the transformation matrices carefully and make sure they correspond to the given equations. They must all be orthogonal matrices.]

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \; ; \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{2}} \\ \frac{-1+\sqrt{3}}{2\sqrt{2}} & \frac{1+\sqrt{3}}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{23}$$

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5. (a) (5 pts) Calculate the arc length in the plane ds^2 for the coordinate system, u, v such that

$$x = u(1 - v) \; ; \quad y = u\sqrt{2v - v^2} \tag{24}$$

.

$$\begin{aligned} ds^2 &= dx^2 + dy^2 = \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)^2 + \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)^2 \\ &= ((1-v)du - u\,dv)^2 + \left((\sqrt{2v - v^2})du + \frac{u(1-v)}{\sqrt{2v - v^2}}dv\right)^2 \\ &= = du^2 + dv^2\left(\frac{u^2}{v(2-v)}\right) \equiv h_u^2 du^2 + h_v^2 dv^2 \end{aligned} \tag{25}$$

(b) (5 pts) Determine the scale factors h_u and h_v, and use them to give an expression for the gradient of a scalar field ψ in these coordinates.

$$h_u = 1 \quad ; \quad h_v = u(2v - v^2)^{-1/2} \tag{26}$$

so

$$\vec{\nabla}\psi = \hat{u}\frac{\partial\psi}{\partial u} + \hat{v}\frac{\sqrt{2v - v^2}}{u}\frac{\partial\psi}{\partial v} \tag{27}$$

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6. (a) (4 pts) Given an arbitrary charge distribution $\rho(\vec{r})$, the electrostatic potential Φ is

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}d\tau'. \tag{28}$$

Find the electrostatic field $\vec{E} = -\vec{\nabla}\Phi$.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\vec{r}\,')\widehat{\vec{r} - \vec{r}\,'}}{(\vec{r} - \vec{r}\,')^2} \tag{29}$$

(b) (3 pts) The charge density of a charge Q spread uniformly over a ring of radius R (centered at the origin, lying in the $x - y$ plane) can be expressed in terms of δ-functions in cylindrical coordinates, $\rho(\vec{r}) = A\delta(r - R)\delta(z)$, where A is a constant. Find A.

Total charge must equal

$$\begin{aligned} Q &= \int d\tau\rho(\vec{r}) = \int_0^\infty r dr \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz\ A\delta(r - R)\delta(z) \\ &= RA \cdot 2\pi \quad \Rightarrow \quad A = \frac{Q}{2\pi R} \end{aligned} \tag{30}$$

(c) (3 pts) Using your result from parts a) and b), evaluate the electric field for the ring of charge at a point $\vec{r}$ on the z axis a distance R above the plane of the ring. [Hint. Think before you calculate: in which direction will $\vec{E}$ point?]

$\vec{E}$ must point in z direction by symmetry. Therefore we are only interested in the z component. For the geometry given $\vec{r} = R\hat{z}$, and $\vec{r}\,' = R\hat{r}$ (cylindrical). Therefore

$$\begin{aligned} \vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\vec{r}\,')\widehat{\vec{r} - \vec{r}\,'}}{(\vec{r} - \vec{r}\,')^2} = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\vec{r}\,')(\vec{r} - \vec{r}\,')}{(\vec{r} - \vec{r}\,')^3} \\ &= \frac{A}{4\pi\epsilon_0}\int_0^\infty r' dr' \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} dz' \delta(r - R)\delta(z)\frac{R\hat{z} - R\hat{r}}{(R^2 + R^2)^{3/2}} \end{aligned} \tag{31}$$

where the denominator has the form shown because $\vec{r} \cdot \vec{r}' = 0$ and $|\vec{r}| = |\vec{r}'| = R$. The $\hat{r}$ term will vanish due to the θ integration, so we keep just the z component as the hint suggests, and perform the integrals over the δ functions, as well as the θ integration, which is trivial:

$$\vec{E}(\vec{r} = R\hat{z}) = \frac{A}{4\pi\epsilon_0 R}\hat{z} = \frac{Q}{8\pi^2\epsilon_0 R^2}\hat{z} \tag{32}$$
