PHY3113–Introduction to Theoretical Physics

Fall 2007

Test 3 – 55 minutes

Oct. 26, 2007

No other materials allowed. If you can't do one part of a problem, solve subsequent parts in terms of unknown answer–define clearly. **Do 4 of 6 problems, clearly indicating which you want graded!**. *You may attempt the remaining two as extra credit as well. All regular parts are worth 10 pts., each extra credit 5 each, for maximum of 50 points.*

Useful formulae:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{1}$$

$$\oint f(z)dz = 0 \tag{2}$$

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz = f(a) \tag{3}$$

$$f^n(a) = \frac{n!}{2\pi i}\oint \frac{f(z)dz}{(z-a)^{n+1}} \tag{4}$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \tag{5}$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}} \quad ; \quad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{-n+1}} \tag{6}$$

1. Two blocks of mass m are connected to springs. The equations of motion can be written as

$$m\ddot{x}_1 = -kx_1 + 2k(x_2 - x_1) \tag{7}$$
$$m\ddot{x}_2 = -kx_2 - 2k(x_2 - x_1) \tag{8}$$

(a) Write the two coupled equations as a matrix equation of the form $|\ddot{X}\rangle = A|X\rangle$, where $|X\rangle$ is a two-component column vector and A is a 2×2 matrix.

(b) Find the eigenvalues and normalized eigenvectors of A.

(c) Draw a picture indicating the motion of the blocks represented by each of the eigenvectors.

2. Consider the 2×2 matrix B with elements $B_{11} = B_{22} = 0$, $B_{12} = 1$ and $B_{21} = -1$.

(a) Show that B rotates any two dimensional vector by $90°$. (Hint: two vectors are at $90°$ with respect to each other if the dot product vanishes.)

(b) Show that $e^{\theta B} = R$, where R is the two dimensional rotation matrix $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.
(Hint: note that $B^2 = -I$, and that $R = \cos\theta I + \sin\theta B$).

3. If $z_1 = \sqrt{2}e^{i\pi/4}$ and $z_2 = (3-i)e^{i\pi}$, evaluate

(a) Re $z_1 z_2$

(b) $\sqrt[3]{z_1 + z_2}$ (give all roots!)

(c) $(z_1 + z_2)^{1+i}$ (hint: use properties of ln)

4. (a) Find the analytic function $w(z) = u(x,y) + iv(x,y)$ if $v(x,y) = 2xy$.

(b) Show explicitly that the function $f(z) = |z|$ is *not* analytic at $z = 0$ by calculating the derivative along 2 different paths.

5. Use contour integration to evaluate the integrals

(a)

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{i+x} dx \tag{9}$$

(b)

$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{1+x^2} dx \tag{10}$$

6. (a) Find the first two a_n terms and first two b_n terms of the Laurent series for $f(z) = (z+2)^{-2}z^{-1}$ around $z = -2$.

(b) Evaluate the residues of $f(z) = (z+2)^{-2}z^{-1}$ at $z = -2$ and $z = 0$.