Test 3 solutions (choose 4/6, + extra credit) **PHZ 3113 Fall 2007**

1. Two blocks of mass m are connected to springs. The equations of motion can be written as

$$m\ddot{x}_1 = -kx_1 + 2k(x_2 - x_1) \tag{1}$$
$$m\ddot{x}_2 = -kx_2 - 2k(x_2 - x_1) \tag{2}$$

(a) Write the two coupled equations as a matrix equation of the form $|\ddot{X}\rangle = A|X\rangle$, where $|X\rangle$ is a two-component column vector and A is a 2×2 matrix.

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \frac{k}{m} \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3}$$

(b) Find the eigenvalues and normalized eigenvectors of A.

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determinant of characteristic matrix is

$$\begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = -5, -1 \tag{4}$$

$$\begin{aligned} \lambda = -5 \quad &\Rightarrow \quad \begin{bmatrix} -3+5 & 2 \\ 2 & -3+5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = -v_2 \\ \lambda = -1 \quad &\Rightarrow \quad \begin{bmatrix} -3+1 & 2 \\ 2 & -3+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = v_2. \end{aligned} \tag{5}$$

Note the eigenvalues are in "units" of k/m, which is just a prefactor of the matrix. Now the normalized eigenvectors are

$$\begin{aligned} \lambda &= -5(k/m): \quad |-5\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \lambda &= -(k/m): \quad |-1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned} \tag{6}$$

Check they're orthogonal:

$$\langle -1|-5\rangle = \frac{1}{2}[1\ 1] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \tag{7}$$

(c) Draw a picture indicating the motion of the blocks represented by each of the eigenvectors.

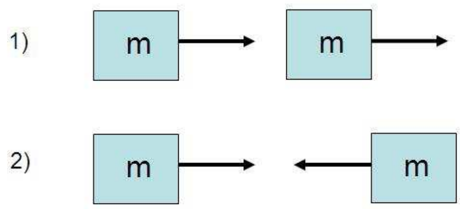

FIG. 1: 1) First eigenvector $\propto (1,1)$ represents uniform motion of both blocks. 2) 2nd represents a "breathing mode" where two blocks oscillate back and forth 180° out of phase.

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2. Consider the 2×2 matrix B with elements $B_{11} = B_{22} = 0$, $B_{12} = 1$ and $B_{21} = -1$.

   (a) Show that B rotates a two dimensional vector by $90°$. (Hint: two vectors are at $90°$ with respect to each other if the dot product vanishes.)

$$B|v\rangle = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ -v_1 \end{bmatrix} = |v'\rangle, \tag{8}$$

so x component has gone into y component, and y comp. into $-x$ comp., $90°$ rotation.

   (b) Show that $e^{\theta B} = R$, where R is the two dimensional rotation matrix $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. (Hint: note that $B^2 = -I$, and that $R = \cos\theta I + \sin\theta B$).

$$\begin{aligned} e^{\theta B} &= 1 + \theta B + \frac{\theta^2 B^2}{2!} + \frac{\theta^3 B^3}{3!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) I + \left(\theta - \frac{\theta^3}{3!} + \dots\right) B \\ &= \cos\theta \cdot I + \sin\theta B \equiv R \end{aligned} \tag{9}$$

where I used the series for sin and cos and the fact that $B^2 = -I$.

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3. If $z_1 = \sqrt{2}e^{i\pi/4}$ and $z_2 = (3 - i)e^{i\pi}$, evaluate

   (a) Re $z_1 z_2$

$z_1 = \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = 1 + i$, and $z_2 = -3 + i$, so $z_1 z_2 = -(4 + 2i)$ and Re $z_1 z_2 = -4$.

   (b) $\sqrt[3]{z_1 + z_2}$

$$\begin{aligned} z_1 &= \sqrt{2}e^{i\pi/4} = \sqrt{2}(\cos\pi/4 + i\sin\pi/4) = 1 + i \\ z_2 &= (3 - i)e^{i\pi} = -3 + i \;\; ; \quad z = z_1 + z_2 = -2 + 2i = \sqrt{8}e^{i\frac{3\pi}{4}} \\ \Rightarrow z^{1/3} &= \sqrt{2}e^{i(\frac{3\pi}{4} + 2m\pi)/3} \end{aligned} \tag{10}$$

   (c) $(z_1 + z_2)^{1+i}$

$$z^{1+i} = e^{(1+i)[\ln\sqrt{8} + i(\frac{3\pi}{4} \pm 2m\pi)]} = \sqrt{8}e^{-(\frac{3\pi}{4} \pm 2\pi m)}e^{i[\ln\sqrt{8} + \frac{3\pi}{4}]} \tag{11}$$

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4. (a) Find the analytic function $w(z) = u(x, y) + iv(x, y)$ if $v(x, y) = 2xy$.

$$v = 2xy \Rightarrow \frac{\partial v}{\partial y} = 2x = \frac{\partial u}{\partial x} \Rightarrow u = x^2 + f(y) \tag{12}$$

$$\frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y} \quad \Rightarrow \quad u = -y^2 + f(x) \quad \Rightarrow u = x^2 - y^2 \tag{13}$$

So

$$w = x^2 - y^2 + i2xy = z^2 \tag{14}$$

(b) Show explicitly that the function $f(z) = |z|$ is *not* analytic at $z = 0$ by calculating the derivative along 2 different paths.

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &\equiv \frac{((z+\Delta z)(z^* + \Delta z^*))^{1/2} - (zz^*)^{1/2}}{\Delta z} \\
&\simeq \frac{(|z|^2 + z\Delta z^* + \Delta z z^*)^{1/2} - (zz^*)^{1/2}}{\Delta z} \\
&= \frac{|z|\left(1 + \frac{\Delta z z^*}{|z|^2} + \frac{\Delta z^* z}{|z|^2}\right)^{1/2} - |z|}{\Delta z} \\
&\simeq \frac{|z|\left(1 + \frac{1}{2}\left[\frac{\Delta z z^*}{|z|^2} + \frac{\Delta z^* z}{|z|^2}\right]\right) - |z|}{\Delta z} \\
&= \frac{\Delta z z^* + \Delta z^* z}{|z|\Delta z}
\end{aligned} \tag{15}$$

Consider now path 1 approaching z: $\Delta y = 0$, $\Delta x \to 0$. Then derivative $\to \mathrm{Re}\, \mathrm{z}/|z|$. On the other hand on path 2 $\Delta x = 0$, $\Delta y \to 0$, derivative $\to -\mathrm{Im}\, z/|z|$. So this is *not* an analytic function at any z.

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5. (a) Use contour integration to evaluate the integrals

i.

$$\int_{-\infty\infty} dx \frac{e^{ix}}{i+x} = \int_C dz \frac{e^{iz}}{i+z}, \tag{16}$$

where C is the real axis, since $z = x$ there. Now note that one can complete the contour in the upper half plane or the lower half plane with an arc at infinity, but only in the upper half plane may we argue that the contribution of the arc is zero. This is because as the Im part of z gets big in the y direction, e^{iz} decays exponentially, $e^{iz} = e^{ix}e^{-y}$. So the integrand is zero on the arc in upper half plane, and we can say that

$$\int_C dz \frac{e^{iz}}{i+z} = \int_{C+arc_+} dz \frac{e^{iz}}{i+z} = 0 \tag{17}$$

The answer zero follows because the integrand is completely analytic in the upper half plane, since the pole at $-i$ is in the lower half plane! Note it doesn't work to complete the contour in the lower half plane since the arc contribution is infinite; can't use Cauchy's theorem.

ii.

$$I = \int_{-\infty}^{\infty} \frac{x \sin \pi x}{1+x^2} dx \tag{18}$$

$$= \mathrm{Im} \int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{1+x^2} dx \quad I' = \oint_C \frac{z e^{i\pi z}}{1+z^2} dz, \tag{19}$$

where C is contour along real axis and closed by an arc at ∞ in upper half plane. Since arc contribution vanishes due to large Im part of z along this part, it can be neglected, and $I = \mathrm{Im} I'$. But

$$I' = 2\pi i R(+i) = i\pi e^{-\pi} \Rightarrow I = \pi e^{-\pi} \tag{20}$$

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6. (a) Find the first two a_n terms and first two b_n terms of the Laurent series for $f(z) = (z+2)^{-2}z^{-1}$ around $z = -2$.

Define $w = z + 2$ and use geometric series:

$$\begin{aligned}
\frac{1}{(z+2)^2}\frac{1}{z} &\equiv \frac{1}{w^2}\frac{1}{w-2} = -\frac{1}{2}\frac{1}{w^2}\frac{1}{1-w/2} \\
&\simeq -\frac{1}{2}\frac{1}{w^2}\left(1 + \frac{w}{2} + \left(\frac{w}{2}\right)^2 + \left(\frac{w}{2}\right)^3 + \dots\right) \\
&= -\frac{1}{2}\frac{1}{(z+2)^2} - \frac{1}{4}\frac{1}{z+2} - \frac{1}{8} - \frac{1}{16}(z+2) + \dots
\end{aligned} \tag{21}$$

(b) Evaluate the residues of $f(z) = (z+2)^{-2}z^{-1}$ at $z = -2$ and $z = 0$.

Once we have the series, the residue is just b_1, the coefficient of $1/(z+2)$, or -1/4.
