Phz3113 Fall '08
Test 3 **solutions**

1. Consider the matrices $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

   (a) (3 pts) Specify whether A and B are i) Hermitian (or antiHermitian); ii) orthogonal; iii) (symmetric (or antisymmetric).

Hermitian means $M = M^\dagger$. $A^\dagger = A^T = -A$, so A is antiHermitian. Orthogonal means $M^{-1} = M^T$: A yes, B yes. A is antisymmetric, i.e. it obeys $M_{ij} = -M_{ji}$. Note B can't be antisymmetric or antiHermitian since its diagonal elements are not zero.

   (b) (3 pts.) Find $C = A + BB^T$.

$$BB^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1}$$

so

$$A + BB^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{2}$$

   (c) (4 pts.) Find eigenvalues and normalized eigenvectors of C. Verify that the eigenvectors are mutually orthogonal.

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \;\Rightarrow\; (1-\lambda)^2 + 1 = 0 \;\Rightarrow\; \lambda = 1 \pm i \tag{3}$$

so for eigenvalues $\lambda = 1 \pm i$ we have

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (1 \pm i) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{4}$$

so

$$\begin{aligned} v_1 + v_2 &= (1 \pm i) v_1 \\ -v_1 + v_2 &= (1 \pm i) v_2 \qquad (5) \\ \Rightarrow \qquad v_2 &= \pm i v_1, \qquad (6) \end{aligned}$$

so eigenvectors are e.g. $\begin{bmatrix} v_1 \\ iv_1 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ -iv_1 \end{bmatrix}$, to normalize divide by magnitude to find $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$. To verify orthogonality remember to complex conjugate the bra vector:

$$\langle 1-i|1+i\rangle = \frac{1}{2}[1\ i]\begin{bmatrix} 1 \\ i \end{bmatrix} = 0 \quad OK \tag{7}$$

---

2. Evaluate the following integrals by contour integration:

   (a) (5 pts.)

$$\int_0^{2\pi} d\theta \, \frac{\sin\theta}{5+3\sin\theta} \tag{8}$$

$$I = \mathrm{Im}\int_0^{2\pi} \frac{e^{i\theta}d\theta}{5+3\sin\theta} \equiv \mathrm{Im}I' \tag{9}$$

Using $z = e^{i\theta}$, $dz = izd\theta$ we find

$$I' = \oint_{C:|z|=1} \frac{dz}{iz}\,\frac{z}{5+\frac{3}{2i}(z-1/z)} = \oint \frac{2zdz}{10iz+3z^2-3} \tag{10}$$

$$= \frac{2}{3}\oint \frac{zdz}{(z+i/3)(z+3i)} = \frac{2}{3}\,2\pi i\,\mathrm{Res}(z=-i/3) = \frac{-i\pi}{6}$$

$$\Rightarrow\ I = -\frac{\pi}{6} \tag{11}$$

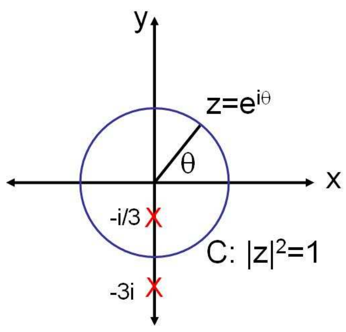

FIG. 1: Prob. 2a

(b) (3 pts.)

$$\int_0^\infty dx \frac{\cos 2x}{9x^2+4} \tag{12}$$

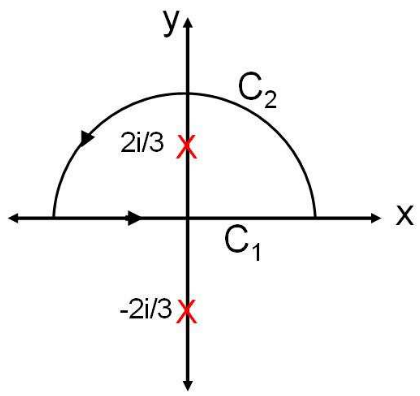

FIG. 2: Prob. 2b

$$I = \int_0^\infty dx \frac{\cos 2x}{9x^2+4} = \frac{1}{2}\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{2ix}dx}{9x^2+4} \equiv \frac{1}{2}\mathrm{Re}\, I_1 \tag{13}$$

Consider

$$I' = \oint_{C=C_1+C_2} \frac{e^{i2z}dz}{9z^2+4} = \oint_C \frac{e^{i2z}dz}{9(z+i\frac{2}{3})(z-i\frac{2}{3})} \tag{14}$$

Only $2i/3$ is enclosed, so

$$I' = \frac{2\pi i}{9} R(2i/3) = \frac{2\pi i}{9}\frac{e^{2i(2i/3)}}{(4i/3)} = \frac{\pi}{6}e^{-4/3}. \tag{15}$$

Since $I = \frac{1}{2}\mathrm{Re}\, I'$, we have

$$I = \frac{\pi}{12}e^{-4/3}. \tag{16}$$

---

3. Given that $w(z) = u(x,y) + iv(x,y)$ is analytic in some region, and that $v(x,y) = e^{-y}\sin x$, find $w(z)$.

$$v = e^{-y}\sin x \ , \quad \frac{\partial v}{\partial x} = e^{-y}\cos x = -\frac{\partial u}{\partial y} \quad \Rightarrow \quad u = \cos x\, e^{-y} + g(x) \tag{17}$$

$$\frac{\partial v}{\partial y} = -e^{-y}\sin x = \frac{\partial u}{\partial x} \quad \Rightarrow \quad u = \cos x e^{-y} + h(y), \tag{18}$$

so only possibility is $u = \cos x e^{-y} + C$, where C is independent of x or y. Dropping the C,

$$w(z) = \cos x e^{-y} + i \sin x e^{-y} = e^{iz} \tag{19}$$

---

4. Given $z_1 = 2e^{i\pi/6}$, and $z_2 = 1 + \sqrt{3}i$, evaluate

(a) $\left|\frac{z_1}{z_2}\right|$

(b) $(z_1 + z_2)^{1/3}$

$z_1 = 2e^{i\pi/6} = \sqrt{3} + i$, $z_2 = 1 + \sqrt{3}i$. The sum is $z_1 + z_2 = (\sqrt{3}+1)(1+i) = \sqrt{2}(\sqrt{3}+1)e^{\frac{i\pi}{4}}$. So a) $\left|\frac{z_1}{z_2}\right| = 1$, and b) $(z_1+z_2)^{1/3} = [\sqrt{2}(\sqrt{3}+1)]^{1/3} e^{i(\frac{\pi}{4}+2m\pi)/3}$.

(c) Find the Fourier transform of a Lorentzian (Γ is real and > 0):

$$f(x) = \frac{\Gamma/\pi}{\mathbf{x}^2 + \Gamma^2} \tag{20}$$

$$\begin{aligned} a(k) &= \frac{1}{\sqrt{2\pi}} \int dx\, e^{-ikx} \frac{\Gamma/\pi}{x^2+\Gamma^2} \\ &= \frac{1}{\sqrt{2\pi}} \int_C dz \frac{e^{-ikz}}{z^2+\Gamma^2}, \end{aligned} \tag{21}$$

where the contour C is along the real z axis and then follows the arc at infinity in the *lower half plane* for $k > 0$. This is because e^{-ikz} is exponentially damped with a factor e^{-ky} in the lower half plane only. Therefore we pick up the pole at $z = -i\Gamma$, get a minus sign from reversing the sign of the contour, and find

$$\begin{aligned} a(k) &= -\frac{1}{\sqrt{2\pi}} 2\pi i R(-i\Gamma) \\ &= -\sqrt{2\pi} i \frac{e^{-k\Gamma}}{-2i\Gamma} = \sqrt{\frac{\pi}{2}} \frac{e^{-k\Gamma}}{\Gamma}, \end{aligned} \tag{22}$$

where we assumed $k > 0$. For $k < 0$ the contour must be closed in the upper half plane, leading to a result $\propto e^{k\Gamma}/\Gamma$. So all together we have

$$a(k) = \sqrt{\frac{\pi}{2}} \frac{e^{-|k|\Gamma}}{\Gamma} \tag{23}$$

5. A mechanical system is described by the set of equations

$$\ddot{x}_1 = x_2 \quad ; \quad \ddot{x}_2 = x_1 \tag{24}$$

where the dot indicates differentiation with respect to time.

(a) Find the normal modes (normalized eigenvectors) of the system.

Eqn. of motion is

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{25}$$

so determinant of characteristic matrix is

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = \pm 1 \tag{26}$$

$$\begin{aligned} \lambda = +1 \quad &\Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = +1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \Rightarrow \quad v_1 = v_2 \\ \lambda = -1 \quad &\Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \Rightarrow \quad v_1 = -v_2. \end{aligned} \tag{27}$$

So normalized eigenvectors are

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad ; \quad |-1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{28}$$

(b) Expand the vector $|V\rangle = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ in terms of these eigenvectors.

$$|v\rangle = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = c_1|1\rangle + c_2|-1\rangle \tag{29}$$

$$c_1 = \langle 1|v\rangle = \frac{7}{\sqrt{2}} \quad ; \quad c_2 = \langle -1|v\rangle = \frac{-3}{\sqrt{2}} \tag{30}$$

So

$$|v\rangle = \frac{1}{\sqrt{2}}(7|1\rangle - 3|-1\rangle). \tag{31}$$

---

6. (Extra credit, 10 pts.) Consider the Pauli matrix $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. Show that $e^{ia\sigma_y} = (\cos a)I + i(\sin a)\sigma_y$, where a is a constant and I is the 2×2 unit matrix.

$$\begin{aligned} e^{ia\sigma_y} &= 1 + a\sigma_y + \frac{(ia\sigma_y)^2}{2!} + \frac{(ia\sigma_y)^3}{3!} + \ldots \\ &= \left(1 - \frac{a^2\sigma_y^2}{2!} + \frac{a^4\sigma_y^4}{4!} + \ldots\right) + i\left(a\sigma_y - \frac{a^3\sigma_y^3}{3!} + \ldots\right) \\ &= \left(1 - \frac{a^2}{2!} + \frac{a^4}{4!} + \ldots\right) + i\sigma_y\left(a - \frac{a^3}{3!} + \ldots\right) \\ &= \cos a \cdot I + i \sin a \, \sigma_y \qquad (32) \end{aligned}$$
